A New Keynesian Model with Diverse Beliefs

by

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Abstract: The paper explores a New Keynesian Model with diverse beliefs and studies the impact of this heterogeneity on fluctuations and monetary policy. It uses a standard model (e.g. Galí (2008), Walsh (2010) and Woodford (2003)). Aggregation is examined only for the log-linearized economy and even for this economy, aggregation problems are significant and their solutions depend upon the belief structure.

Agents' beliefs are described by individual state variables and satisfy three Rationality Axioms. Belief rationality plays a key role in driving belief dynamics and mean market belief is the main tool used to solve the aggregation problems. Macro dynamics is then described by an IS curve, Phillips curve and a monetary rule similar to standard models except that mean market belief is a new force amplifying fluctuations. Due to belief heterogeneity, changes in the policy rule alter key macro-economic parameters which must be deduced from the micro equilibrium, a problem not present in a single agent economy. In addition to belief rationality agents know the equilibrium map.

Diverse beliefs alter the problem faced by a central bank since a central source of fluctuations in this paper are not exogenous shocks (assumed small) but fluctuations caused by market expectations, and this alters the role of a central bank. Diverse beliefs impact response to policy due to their effect on motives to consume, supply labor etc. but market belief may support or oppose a central bank's goals. The paper draws general conclusion about efficacy of monetary rules of either contemporaneous or of expected output deviation and inflation with weights (ξ_y, ξ_π). The main conclusions are as follows. (i) Monetary policy can counter the effects of market belief by aggressive anti-inflation ξ_π policy, but with cost. Large ξ_π entails volatile financial markets and volatile individual consumption. Volatility of aggregate output is different from volatility of individual consumption and welfare considerations suggest that individual consumption volatility and financial market instability are at least as important goals of central bank as stability of aggregate output. The paper then shows that optimal policy outcomes require a central bank to employ moderate values of ξ_π. (iii) A central bank that aims to stabilize only inflation and aggregate output can be either a one mandate bank that fights only inflation or a two mandate central bank: each can attain a different segments of the efficiency frontier. (ii) Due to diverse beliefs, the effects of (ξ_y, ξ_π) are not monotonic. (iii) As a result of (ii) the problem of output stabilization is particularly complex. Indeed, response monotonicity is a desirable property, offering a central bank feasible policy actions whose outcomes are predictable and entailing clearer policy trade-off.(iv) Both efficiency and monotonicity of output stabilization are improved if a central bank uses rules that target inflation and the *causes of output volatility which are market belief and exogenous shocks, instead of output.* Targeting market belief may be accomplished by targeting asset prices, which reflect market belief.

As to optimal policy and "forecast-targeting" it is seen that under diverse beliefs a bank's optimal policy is not Pareto Optimal and may not even be Pareto improving. Central bank's policy does not alter agents' beliefs about state variables and under forecast targeting the private sector does not adopt the central bank's forecasts even when agents fully understand and accept the policy commitment of the bank since the bank and private sector may disagree about forecasts of endogenous variables. Hence, a bank's optimal policy may be carried out with private market opposition rather than consensus, as is the case under Rational Expectations. Also, an optimal policy relative to a bank's belief adds to privately perceived uncertainty of future bank's belief or actions even when the policy is fully understood.

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A New Keynesian Model withe Diverse Beliefs[2]

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The New Keynesian model has become an important tool of macroeconomics. Due to its assumption of monopolistic competition, prices are firms' strategic variables and price stickiness a cause for money non-neutrality and efficacy of monetary policy. The model is inherently heterogenous since the large number of household-firms produce different intermediate goods. It is thus only natural to ask what is the effect of heterogeneity on the conduct of monetary policy. This paper formulates a model of the New Keynesian theory with diverse beliefs, aiming to investigate whether diversity of beliefs matters to macro dynamics and if it does, what are its implications for the conduct of monetary policy. In examining such implications I comment on optimal monetary policy, but the paper focuses on the *impact of diverse beliefs on feasible outcomes of different policy rules.* The paper consists of two parts. In the first I start from the underlying microeconomic structure and ask whether an aggregate macroeconomic New Keynesian model can be constructed in a consistent manner by aggregating the micro economy. An aggregate model is a system of difference equations among economic aggregates whose solution traces the equilibrium evolution of the aggregates. In the second part I study the monetary policy implications of belief heterogeneity.

Before proceeding I note that as the era of Rational Expectations (in short RE) comes to a close, it is useful to keep in mind two points. First, the success of RE in disciplining macroeconomic modeling should not obscure the fact that the term "rational" is merely a label. Rationality of actions and rationality of beliefs have little to do with each other and using the term "rational" in RE has tended to brand all other beliefs as "irrational." Rational agents who hold diverse beliefs do not satisfy the RE requirements but may satisfy other plausible principles of rationality. Indeed, the study of axioms of belief rationality is a fruitful area of research that can fill the wide open space between the extremities of RE and true irrational beliefs.

A second point relates to private information. To avoid contradicting RE, many use the device of asymmetric private information as the "cause" of diverse beliefs. Indeed, some view diverse beliefs as

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equivalent to asymmetric information. This is theoretically and empirically the wrong solution and in Kurz (2008), (2009) I explain why. Suffices to say that market behavior of agents holding diverse beliefs with common information is very different from the case when they have private information. Under private information individuals guard their private information and deduce private information from prices. Without private information agents are willing to reveal their forecasts and use the opinions of others (i.e. market belief) only to forecast future prices and other endogenous variables, not as a source from which to deduce information they do not have. In addition, all empirical evidence associate diverse forecasts to diverse modeling or diverse interpretation of public information. Finally, the volatility of RE models with private information are fully determined by exogenous shocks, consequently they cannot deliver the main dynamic implications of economies with rational and diverse beliefs with common information (see Kurz (2009)). The key implication is that diverse beliefs constitute a volatility amplification mechanism hence excess economic fluctuations are caused by diverse beliefs and this added economic risk is generated *within the economy,* not by exogenous shocks. I have called it (Kurz (1997)) *Endogenous Uncertainty*. These dynamic properties are explored in Kurz (2009), (2011) and briefly discussed later in Section 4.

To explore problems of aggregating a microeconomic model to construct a macro model, I concentrate on the standard version of the New Keynesian theory. With this in mind I follows developments in Woodford (2003), Walsh (2010) and Gali (2008). I note the axiomatic approach of Branch and McGough (2009), a method adopted by others such as Branch and Evans (2006), (2011) and Branch and McGough (2011). The Branch and McGough's (2009) axioms are made directly on the expectation operators, not on beliefs. As they are motivated by bounded rationality, they violate typical models with diverse beliefs. In contrast, I specify *rationality* axioms on beliefs and show they offer a natural route to a New Keynesian model with diverse beliefs in which aggregation is attained in the log linearized economy. This last point is important since it will be clear a "representative household" does not exist in the model developed below and aggregation of the true economy is not possible in most cases. Instead, I study the aggregation problem *in the log linear economy* which is the standard economy used for virtually any policy analysis.

Ideas about diverse beliefs used in this paper are drawn from a long literature. Kurz (1994), (1997) offer an early collection of papers and Kurz (2009), (2011) are recent surveys. As to business cycles and monetary policy, Motolese (2001),(2003) shows that diverse beliefs cause, on their own, money non-neutrality. Kurz, Jin and Motolese (2005) and Jin (2007) offer the first formal models showing diverse beliefs constitute an independent cause for business cycle fluctuations and model calibration that reproduces the observed data of the US economy. In the same spirit Branch and McGough (2011) and De Grauwe (2011) show that boundedly rational diverse beliefs cause an amplification of business cycle fluctuations. Other approaches to the problem include Lorenzoni (2009) and Milani (2011). Nielsen (2003), (2011) studies Pareto improving policies that stabilize economic volatility caused by dynamics of diverse belief with the use of taxes and subsidies to counteract the effect of beliefs.

As to the impact on monetary policy, the Kurz, Jin and Motolese (2005) model have two types of infinitely lived dynasties with diverse beliefs but full price flexibility and money non neutrality, showing that sticky prices offer only one of the routes to efficacy of monetary policy. They investigate the ability of different monetary policy rules to stabilize fluctuations caused primarily by diverse beliefs. That is, in the Kurz, Jin and Motolese (2005) model belief diversity is a volatility amplification mechanism which, in turn, becomes the object of monetary stabilization policy. Other non RE papers that study efficacy of policy, approach it from the perspective of learning. Howitt (1992) uses a standard macroeconomic model and shows instability under learning of interest rate pegging and related rules. Similarly, Bullard and Mitra (2002) show that if agents follow adaptive learning, the stability of the Taylor-type rules is questionable. Evans and Honkapohja (2003),(2006) study a New Keynesian model with a representative agent but with non RE belief due to learning. They study the joint stability of the economy and learning and show convergence to RE under stability of learning. They assume agents are boundedly rational as they do not know the equilibrium map and make forecasts based only on their learning model. I explicitly assume agents are rational and know the equilibrium map but have diverse beliefs about the state variables of the system.

What are the paper's results? Sections 1- 5 explore the problem of aggregating the log linearized economy and show that, under the assumed structure of belief, a consistent macroeconomic modes results from such aggregation. However, the aggregate model has key parameters deduced from the microeconomic equilibrium and which, in turn, depend upon the policy parameters. Hence, if a policy rule is changed, these equilibrium parameters need to be derived from the new microeconomic equilibrium and consequently the macro model itself changes. Hence, part of evaluating feasible stabilization of a policy rule is the rule's impact on the parameters of the macroeconomic model it induces. This process of evaluation is entirely absent from the standard macroeconomic model based on the representative agent.

Section 6 is devoted to a simulation study of the impact of diverse beliefs on efficacy of monetary policy. Understanding the results requires a clarification of what the central bank aims to stabilize and what is the perspective offered by a theory of fluctuations when agents hold diverse beliefs. A standard Real

Business Cycles (RBC) model assumes that technology shocks (to be defined later) have a standard deviation of 0.0072 deduced from the Solow residual, a practice that has been universally rejected, leading to a consensus that the true standard deviation is much smaller. My starting point is therefore a value of 0.003 assigned to this standard deviation with the important implications that *much of the model's volatility is due to the effect of expectations and beliefs*. This is a crucial change in the problem faced by a central bank since the first main conclusion of this paper is that an *efficient policy rule depends upon the nature of the shocks and the cause of volatility*. Comparisons between the results of this paper with standard results in the literature show that the central bank has different tasks in the two models: in a standard RBC model under RE the driving force is a large technology shock (perhaps with other exogenous shocks) whose effect the central bank aims to stabilize under conditions of price stickiness. In the models of this paper the exogenous shocks are small and the central bank is concerned with stabilizing the large effect of market expectations on economic volatility as well as with the consequences of price stickiness. Hence, the main objective is to explore how efficient central bank policy changes when the central bank is faced with such new conditions under which policy is responsive to the volatility impact of market expectations.

Volatility effects of expectations is clearly present even in models with flexible prices, hence a central bank must stabilize the volatility of *actual* output level (or its deviation from steady state) rather than the volatility of the gap between output and the output level at the flexible price equilibrium. Section 6.1 provides details on why the gap is not an object of central bank stabilization. A comparison of the results of stabilization reported here with other models should keep this fact in mind.

As to stabilization, I study the impact of belief diversity on feasible outcomes of policy rules which are defined on inflation and output or on expected inflation and output, with weights ξ_π and ξ_y, respectively. Under a standard RE formulation the response is monotonic in the two instruments: output volatility *falls* with ξ_y and *rises* with ξ_π while inflation volatility *rises* with ξ_y and *falls* with ξ_π. This monotonicity implies a tradeoff according to which a central bank faces a policy choice between volatility of output and inflation although in some cases being aggressive on both counts improves the bank's performance. I first show that even under RE this tradeoff depends upon the nature of the shocks and a large shock in the IS curve can eliminate the tradeoff. With diverse beliefs and other exogenous shocks these monotonicity results do not hold. Efficient policy varies with the stipulated exogenous shocks but I outline three distinct results:

(i) The effect of policy instruments (ξ_y, ξ_π) on volatility of inflation and output is not monotonic, leading the policy space to split into two sub-regions, which depend upon the model's shocks. For the shocks of this

paper the two regions are as follows. In the first the central bank acts as a single mandate bank, aggressively fighting inflation by using as large ξ_π weight as politically feasible and setting $\xi_y = 0$. In this region the policy will, in fact, reduce the volatility of both inflation and output but in this region output volatility cannot be reduced below some lower bound. In the second region the central bank act as a dual mandate central bank and the policy space offers a tradeoff between inflation volatility and output volatility. Both regions are part of the efficient set of policy outcomes.
(ii) As a result of (i) the *output stabilization* problem is complex. Due to diverse beliefs a central bank faces a new problem which arises from the fact that mean output (which equals mean consumption) is not equal to individual consumption and the volatility of mean consumption is very different from the volatility of individual consumption. Although an important policy tradeoff allows a central bank choice of policies in the first or second regions of the policy space in (i), regardless of the policy selected, any policy that stabilizes the volatility of output and inflation typically entails highly volatile individual consumption. Hence, in an economy with diverse beliefs a central bank must balance off three goals: inflation, output volatility and individual consumption volatility. Since the volatility of individual consumption is associated with volatility of the real rate, this volatility is actually the volatility of financial markets. In all cases there is a clear tradeoff between inflation volatility and individual consumption volatility.
(iii) Efficiency and monotonicity of policy outcomes is improved if a central bank uses a different rule to target inflation and the *causes of output volatility (i.e. state variables) instead of output.* State variables in this paper are: a technology shock, a policy shock and, most important, market belief (to be explained). Such policy rule typically has a monotonic effect on volatility. It reacts to the forces that cause output fluctuations rather than to fluctuations of output. Given the clear tradeoff between inflation and individual consumption volatility it is seen that in that tradeoff aggressive anti-inflation ξ_π policies are not desirable and instead, efficient policies are typically moderate.

Finally, in Section 7 I examine the problems arising from using forward looking rules, optimal monetary policy and forecast targeting in an economy with diverse beliefs. The general style of the paper is an exploration of the basic ideas while leaving many open issues for future research.

1. Household j's Problem and Euler Equations

The standard formulation starts with a continuum of agents and products but this formulation is not natural when one draws a random sample of the order of the continuum. Hence, although in the development below I write integrals for mean values, it is natural to think of such integrals as arising in a large economy when one takes limits of means as sample size increases to infinity.

Household j is a producer-consumer that produces intermediate commodity j at price p_{jt} with production technology which uses only labor without capital defined by

$$Y_{jt} = \zeta_t N_{jt} \quad , \quad \zeta_t > 0 \text{ a random variable with } E^m(\zeta_t) = 1 .$$

I explain later what the probability measure m is. The household solves a maximization problem with a penalty on excessive borrowing and lending of the form

$$\text{(1) Max } E_t^j \sum_{\tau=0}^{\infty} \beta^\tau \left(\frac{1}{1-\sigma}(C_{t+\tau}^j)^{1-\sigma} - \frac{1}{1+\eta}(L_{t+\tau}^j)^{1+\eta} + \frac{1}{1-b}(\frac{M_{t+\tau}^j}{P_{t+\tau}})^{1-b} - \frac{\tilde{\tau}_b}{2}(\frac{B_{t+\tau}^j}{P_{t+\tau}})^2 \right), 0<\beta<1, \sigma>0, \eta>0, b>0.$$

The penalty replaces an institutional constraint. I set $\tilde{\tau}_b$ very small with the view to implement transversality conditions, considering a solution with explosive borrowing to be a non-equilibrium. The budget constraint, with transfers used for redistribution to be explained below, is defined by

$$\text{(2a) } C_t^j + \frac{M_t^j}{P_t} + \frac{B_t^j}{P_t} + \frac{T_t^j}{P_t} = (\frac{W_t}{P_t})L_t^j + [\frac{B_{t-1}^j(1+r_{t-1}) + M_{t-1}^j}{P_{t-1}}](\frac{P_{t-1}}{P_t}) + \frac{1}{P_t}[p_{jt}Y_{jt} - W_t N_{jt}] \ , \ (\frac{P_t}{P_{t-1}}) = \pi_t$$

(2a) (M_0^j , B_0^j) is given, all j. Initial aggregate debt is 0 and aggregate money supply at t = 0 is given.

C is consumption, L is labor, M is money holding, T are transfers, W is nominal wage, B is borrowing and r is a nominal interest rule defined as a function of aggregate variables specified later. Equilibrium real balances, inflation rate and nominal interest rate will then determine the equilibrium price level.

The standard Euler equations are as follows. Optimum with respect to bond purchases B_t^j is

$$\text{(3a)} \qquad \tilde{\tau}_b(\frac{B_t^j}{P_t}) + (C_t^j)^{-\sigma} = E_t^j[\beta \left(C_{t+1}^j\right)^{-\sigma} \frac{1+r_t}{\pi_{t+1}}] .$$

Optimum with respect to labor is

$$\text{(3b)} \qquad (C_t^j)^{-\sigma}(\frac{W_t}{P_t}) = (L_t^j)^\eta$$

and optimum with respect to money is

$$\text{(3c)} \qquad 1 - \frac{(M_t^j/P_t)^{-b}}{(C_t^j)^{-\sigma}} = E_t^j[\beta \left(\frac{C_{t+1}^j}{C_t^j}\right)^{-\sigma} \frac{1}{\pi_{t+1}}] .$$

(3a)-(3c) imply that the demand for money is determined by the following condition

$$\text{(4)} \qquad \frac{(M_t^j/P_t)^{-b}}{(C_t^j)^{-\sigma}} = \frac{r_t}{1+r_t} - \frac{\tilde{\tau}_b}{1+r_t}(\frac{B_t^j}{P_t(C_t^j)^{-\sigma}}).$$

I proceed as in a cashless economy by ignoring (4) and how the central bank provides liquidity to satisfy the demand for money in (4) via the agent's transfers. The central bank sets the nominal interest rate.

I now log linearize the Euler equations. If X has a riskless steady state $\bar{X}$ then the notation is $\hat{x}_t = \log(X_t/\bar{X})$ except for borrowing when $\hat{b}_t = B_t/(P_t\bar{Y})$ with a zero steady state value. I then have log linear approximations with the zero inflation steady state hence I let $\bar{\pi} = 1$, $\log(P_t/P_{t-1}) = \hat{\pi}_t$:

(5a) $$\hat{c}_t^j = E_t^j(\hat{c}_{t+1}^j) - (\frac{1}{\sigma})[\hat{r}_t - E_t^j(\hat{\pi}_{t+1})] + \tau_b\hat{b}_t^j \quad , \tau_b = \frac{\tilde{\tau}_B}{\sigma}\bar{Y}^{1+\sigma} \quad , \tau_b < 10^{-4}$$

(5b) $$-\sigma(\hat{c}_t^j) + (\hat{w}_t - \hat{p}_t) = \eta(\hat{\ell}_t^j).$$

In steady state $\bar{C}^j = \bar{Y}^j = \bar{C} = \bar{Y}$, $\bar{\pi} = 1$, $\bar{L}^j = \bar{N}^j = \bar{Y}$. The final term $\tau_b\hat{b}_t^j$ imposes j's transversality conditions which insists on bounded borrowing. Define the aggregate variables

$$\hat{c}_t = \int_0^1 \hat{c}_t^i di \quad , \quad \hat{y}_t = \int_0^1 \hat{y}_t^i di \quad , \quad \hat{n}_t = \int_0^1 \hat{n}_t^i di \quad , \quad \hat{\ell}_t = \int_0^1 \hat{\ell}_t^i di \quad , \quad \hat{b}_t = \int_0^1 \hat{b}_t^i di .$$

Now observe that (5b) aggregates and the equilibrium conditions $\hat{c}_t = \hat{y}_t$, $\hat{n}_t = \hat{\ell}_t$ imply the relation

(5b') $$(\hat{w}_t - \hat{p}_t) = \eta(\hat{n}_t) + \sigma(\hat{y}_t).$$

On the other hand, (5a) does not aggregate since it entails an expression of the form

$$\int_0^1 E_t^j(\hat{c}_{t+1}^j)dj \quad \text{or in the finite case} \quad \frac{1}{N}\sum_{j=1}^{N} E_t^j(\hat{c}_{t+1}^j).$$

Average individuals' forecasts of the deviation of their future consumption from steady state is computable number but is not a natural macro economic aggregate. For this reason I first rewrite (5a) as

(5a') $$\hat{c}_t^j = E_t^j(\hat{c}_{t+1}) + \left(E_t^j(\hat{c}_{t+1}^j) - E_t^j(\hat{c}_{t+1})\right) - (\frac{1}{\sigma})[\hat{r}_t - E_t^j(\hat{\pi}_{t+1})] + \tau_b\hat{b}_t^j.$$

Next, introduce

Definition 1: $\bar{E}_t = \int_0^1 E_t^j dj$ means: for any random variable X, $\bar{E}_t(X) = \int_0^1 E_t^j(X)dj$.

Average agents' diverse probabilities is not a proper probability and the operator $\bar{E}_t$ is not a conditional expectation deduced from a probability measure (see Kurz (2008)). It is an average forecast and does not obey the law of iterated expectations. Since in equilibrium $\hat{c}_t = \hat{y}_t, \hat{b}_t = 0$, averaging (5a') leads to

(6) $$\hat{y}_t = \bar{E}_t(\hat{y}_{t+1}) + \int_0^1 \left(E_t^j(\hat{c}_{t+1}^j) - E_t^j(\hat{c}_{t+1})\right)dj - (\frac{1}{\sigma})[\hat{r}_t - \bar{E}_t(\hat{\pi}_{t+1})] .$$

Individual penalties vanish while the middle term does not aggregate. It occurs when mean agents' forecasts of *own* consumption differ from mean forecast of *mean* consumption. In (6) I use the definition

(6a) $$\Phi_t(\hat{c}) = \int_0^1 \left(E_t^j(\hat{c}_{t+1}^j) - E_t^j(\hat{c}_{t+1})\right)dj.$$

<u>Proposition 1:</u> Under diverse beliefs the IS curve in a log linearized economy is defined by (6)-(6a)

(7) $$\hat{y}_t = \bar{E}_t(\hat{y}_{t+1}) + \Phi_t(\hat{c}) - (\frac{1}{\sigma})[\hat{r}_t - \bar{E}_t(\hat{\pi}_{t+1})]$$

where the term $\Phi_t(\hat{c})$ is not directly aggregated. It reflects the structure of market belief.

Diverse beliefs has thus a dual impact on (7): the mean forecast operator $\bar{E}_t$ which violates the law of iterated expectations and the term $\Phi_t(\hat{c})$. Under RE and representative household $E_t^j(\hat{c}_{t+1}^j) = \bar{E}_t(\hat{c}_{t+1})$ and the extra terms disappear. These terms are natural to diverse beliefs hence pivotal issues to be examined.

2. Demand function of agent j for consumption under monopolistic competition

I adopt a standard model of household-producer-monopolistic competitor with the Calvo (1983) model for sticky prices hence the development is familiar. There is a large number (perhaps a continuum or, equivalently, a large N) of products and each agent produces one product which is substitutable with all others. Final consumption of household j is constructed from intermediate outputs as follows:

$$C_t^j = [\int_0^1 (C_{it}^j)^{\frac{\theta-1}{\theta}} di]^{\frac{\theta}{\theta-1}} \ , \ \theta > 1.$$

At price p_{it} consumption cost is $\int_0^1 p_{it} C_{it}^j di$. Minimizing cost subject to $C_t^j \leq [\int_0^1 (C_{it}^j)^{\frac{\theta-1}{\theta}} di]^{\frac{\theta}{\theta-1}}$ leads to

(8) $$C_{it}^j = (\frac{p_{it}}{\psi_t})^{-\theta} C_t^j$$

P_t is price of final consumption, which is the price level. Equilibrium in the final goods market requires

(8a) $$P_t \equiv [\int_0^1 p_{it}^{1-\theta} di]^{\frac{1}{1-\theta}}$$

Aggregate (8) over households j to obtain the market demand function for intermediate commodity i, given aggregate consumption. But aggregate consumption equals aggregate income. Hence, considering j who produces intermediate good j, the demand for firm's j product is defined by

(8b) $$Y_{jt}^d = (\frac{p_{jt}}{P_t})^{-\theta} Y_t \text{ with implied required labor input of } N_{jt} = \frac{1}{\zeta_t}(\frac{p_{jt}}{P_t})^{-\theta} Y_t.$$

With probability (1- ω) a firm adjusts prices at each date, independently over time.

Key Assumption 1: In a Calvo model firms with diverse beliefs select different optimal prices. *Assume that the sample of firms allowed to adjust prices at each date is selected independently across agents hence the distribution of agents in terms of output or belief is the same whether one looks at those who adjust prices or those who do not adjust prices.*

I now examine the price level in (8a). At t a random sample is taken as a set of firms S_t in [0,1] of measure $1-\omega$ that adjust prices at t and S_t^C in [0,1] of measure ω that do not adjust. By the key Assumption 1 the mean price of those firms that do not change price equals the date t-1 price hence

$$P_t^{1-\theta} = \int_{S_t} p_{jt}^{\star(1-\theta)} dj + \int_{S_t^C} p_{j,t-1}^{(1-\theta)} dj = \int_{S_t} p_{jt}^{\star(1-\theta)} dj + \omega P_{t-1}^{1-\theta}.$$

$p_{jt}^{\star}$ is the optimal price of j hence,

(9) $$1 = \int_{S_t} \left(\frac{p_{jt}^{\star}}{P_t}\right)^{(1-\theta)} dj + \omega \left(\frac{P_{t-1}}{P_t}\right)^{1-\theta}.$$

Define $q_{jt}^{\star} = \frac{p_{jt}^{\star}}{P_t}$ and log linearize (9) to conclude the equation $0 = \int_{S_t} \hat{q}_{jt}^{\star} dj - \omega \hat{\pi}_t$. Hence I have

(10a) $$\int_{S_t} \hat{q}_{jt}^{\star} dj = \omega \hat{\pi}_t.$$

At steady state $\bar{p}_j = \bar{P}$, and using notation $\Delta X_t = X_t - \bar{X}$, it follows from (9) that a log linearization leads to

(10b) $$\int_{S_t^C} \Delta\left(\frac{p_{j,t-1}}{P_t}\right) dj = -\omega \hat{\pi}_t$$

By Assumption 1, with probability 1, (10a) is independent of sets S_t. The distributions of characteristics are the same in all random sets and (10a) changes only by change in state variables of the economy. If every firm selects its optimal price, the mean over the population is related to (10a) through the relation

$$\int_{S_t} \hat{q}_{jt}^{\star} dj = (1-\omega) \int_0^1 \hat{q}_{jt}^{\star} dj \quad \Rightarrow \quad \int_0^1 \hat{q}_{jt}^{\star} dj = \frac{\omega}{1-\omega} \hat{\pi}_t .$$

Marginal Cost. Since $Y_{jt} = \zeta_t N_{jt}$ variable cost function of j is $W_t \frac{Y_{jt}}{\zeta_t}$. Nominal marginal cost is $\frac{W_t}{\zeta_t}$ and real marginal cost is $\varphi_t = \frac{1}{\zeta_t} \frac{W_t}{P_t}$. Deviations from steady state are therefore $\hat{\varphi}_t = -\hat{\zeta}_t + \hat{w}_t - \hat{p}_t$.

Since agent j is a monopolistic competitor, maximizing (1) with respect to output is the same as maximizing with respect to p_{jt}. In the next section I use the demand function to define the profits function:

(11) $$\Pi_{jt} = \frac{1}{P_t}[p_{jt} Y_{jt} - W_t N_{jt}] = \left[\frac{p_{jt}}{P_t} - \frac{1}{\zeta_t} \frac{W_t}{P_t}\right] Y_{jt} = \left[\left(\frac{p_{jt}}{P_t}\right)^{1-\theta} - \frac{1}{\zeta_t} \frac{W_t}{P_t} \left(\frac{p_{jt}}{P_t}\right)^{-\theta}\right] Y_t$$

3. Optimal Pricing of intermediate goods

Agent j owns firm j and manages its business. His optimal pricing is selected by maximizing (1) subject to (2) and (11) together with the Calvo type price limitation.

Insurance and Anonymity Assumption 2: An agent-firm chooses an optimal price subject to the budget constraint (2) and (11) and *considers the transfer as a lump sum*. However, the level of transfers received ensures all firms have the same real profits. Hence, transfers to firm j equal

$$\frac{T_t^j}{P_t} = \Pi_t - \Pi_t^j \quad , \quad \Pi_t = \int_0^1 \Pi_t^j dj .$$

Discussion: Assumption 2 removes all income effects of random price adjustments. It is equivalent to assuming either that profits are insured or that all agents-firms have equal ownership share in all firms but agent-firm j manages firm j by selecting an optimal price so as to maximize (1) subject to (2). Anonymity means here that agent-firm j assumes it is small and has no effect on the transfers it receives or pays.

Profit in (11) requires j to select optimal price to maximize (1) subject to the budget constraint at all future dates ($t+\tau$) in which, with probability ω^τ, the firm cannot change the price at t. The budget is

$$C_{t+\tau}^j + \frac{M_{t+\tau}^j}{P_{t+\tau}} + \frac{B_{t+\tau}^j}{P_{t+\tau}} + \frac{T_{t+\tau}^j}{P_{t+\tau}} = (\frac{W_{t+\tau}}{P_{t+\tau}})L_{t+\tau}^j + [\frac{B_{t+\tau-1}^j(1+r_{t+\tau-1}) + M_{t+\tau-1}^j}{P_{t+\tau-1}}](\frac{P_{t+\tau-1}}{P_{t+\tau}}) + [\frac{p_{jt}}{P_{t+\tau}} - \frac{1}{\zeta_{t+\tau}}\frac{W_{t+\tau}}{P_{t+\tau}}](\frac{p_{jt}}{P_{t+\tau}})^{-\theta}Y_{t+\tau}.$$

Now, the first order conditions apply only to terms involving $p_{jt}^\star$ and these conditions are: $Y_{jt}^d = (\frac{p_{jt}}{P_t})^{-\theta}Y_t$

$$E_t^j\sum_{\tau=0}^{\infty} \beta^\tau\omega^\tau(C_{t+\tau}^j)^{-\sigma}\left((1-\theta)(\frac{p_{jt}^\star}{P_{t+\tau}})^{-\theta} + \theta\varphi_{t+\tau}(\frac{p_{jt}^\star}{P_{t+\tau}})^{-\theta-1}\right)\frac{Y_{t+\tau}}{P_{t+\tau}} = 0$$

where $\varphi_{t+\tau} = \frac{1}{\zeta_{t+\tau}}\frac{W_{t+\tau}}{P_{t+\tau}}$. Using (8b) this condition is equivalent to

$$\left[E_t^j\sum_{\tau=0}^{\infty} \beta^\tau\omega^\tau(C_{t+\tau}^j)^{-\sigma}Y_{t+\tau}(\frac{P_t}{P_{t+\tau}})^{-\theta}\left((1-\theta)(\frac{p_{jt}^\star}{P_t})(\frac{P_t}{P_{t+\tau}}) + \theta\varphi_{t+\tau}\right)\right]\frac{1}{p_{jt}^\star}(\frac{p_{jt}^\star}{P_t})^{-\theta} = 0.$$

Cancel the end terms and solve for $(\frac{p_{jt}^\star}{P_t})$ to deduce the optimal price of a firm that adjusts price at date t

$$(12) \qquad (\frac{p_{jt}^\star}{P_t}) = (\frac{\theta}{\theta-1})\frac{E_t^j\sum_{\tau=0}^{\infty} \beta^\tau\omega^\tau(C_{t+\tau}^j)^{-\sigma}Y_{t+\tau}\varphi_{t+\tau}(\frac{P_{t+\tau}}{P_t})^{\theta}}{E_t^j\sum_{\tau=0}^{\infty} \beta^\tau\omega^\tau(C_{t+\tau}^j)^{-\sigma}Y_{t+\tau}(\frac{P_{t+\tau}}{P_t})^{\theta-1}}.$$

Aiming to aggregate (12) I log linearize it as follows. First write it as

$$\sum_{\tau=0}^{\infty} \beta^\tau\omega^\tau E_t^j[(C_{t+\tau}^j)^{-\sigma}Y_{t+\tau}(\frac{P_{t+\tau}}{P_t})^{\theta-1}]q_{jt}^\star = (\frac{\theta}{\theta-1})\sum_{\tau=0}^{\infty} \beta^\tau\omega^\tau E_t^j[(C_{t+\tau}^j)^{-\sigma}Y_{t+\tau}\varphi_{t+\tau}(\frac{P_{t+\tau}}{P_t})^{\theta}].$$

Log linearization of the left hand side around the riskless steady state yields

$$\frac{(\bar{C}^{j})^{1-\sigma}}{(1-\beta\omega)} + \frac{(\bar{C}^{j})^{1-\sigma}}{(1-\beta\omega)}\hat{q}^{\star}_{jt} + (\bar{C}^{j})^{1-\sigma}\sum_{\tau=0}^{\infty} \beta^{\tau}\omega^{\tau}E_t^j[-\sigma(\hat{c}^j_{t+\tau}) + \hat{y}_{t+\tau} + (\theta-1)(\hat{p}_{t+\tau}-\hat{p}_t)]$$

Log linearization of the right hand side leads to

$$(\frac{\theta}{\theta-1})\left(\frac{(\bar{C}^{j})^{1-\sigma}}{(1-\beta\omega)}\bar{\varphi} + \bar{\varphi}(\bar{C}^{j})^{1-\sigma}\sum_{\tau=0}^{\infty} \beta^{\tau}\omega^{\tau}E_t^j[\hat{\varphi}_{t+\tau} - \sigma(\hat{c}^j_{t+\tau}) + \hat{y}_{t+\tau} + \theta(\hat{p}_{t+\tau}-\hat{p}_t)]\right).$$

Equalizing both note two facts. First, in the steady state prices are flexible and it is well known that

$$(\frac{\theta}{\theta-1})\bar{\varphi} = 1.$$

Second, when equalizing the two sides all terms involving $(\bar{C}^{j})^{1-\sigma}$, $E_t^j(\hat{c}^j_{t+\tau})$ and $\hat{y}_{t+\tau}$ cancel and I have

(13) $$\frac{\hat{q}^{\star}_{jt}}{(1-\beta\omega)} = \sum_{\tau=0}^{\infty} \beta^{\tau}\omega^{\tau}E_t^j[\hat{\varphi}_{t+\tau} + (\hat{p}_{t+\tau}-\hat{p}_t)] = \sum_{\tau=0}^{\infty} \beta^{\tau}\omega^{\tau}E_t^j[\hat{\varphi}_{t+\tau} + \hat{p}_{t+\tau}] - \frac{1}{(1-\beta\omega)}\hat{p}_t.$$

(14)

(13) shows the only difference among firms that adjust prices arises due to difference in expectations of economy wide variables. No j specific variable appears on the right side. From (13) one deduces that

$$\hat{q}^{\star}_{jt} + \hat{p}_t = (1-\beta\omega)[\hat{\varphi}_t + \hat{p}_t] + \beta\omega E_t^j\left((1-\beta\omega)\sum_{\tau=0}^{\infty} \beta^{\tau}\omega^{\tau}E^j_{t+1}[\hat{\varphi}_{t+1+\tau} + \hat{p}_{t+1+\tau}]\right).$$

It leads to a relation between optimal price at t and expected optimal price at $t+1$ if j can adjust price at $t+1$:

$$\hat{q}^{\star}_{jt} + \hat{p}_t = (1-\beta\omega)[\hat{\varphi}_t + \hat{p}_t] + \beta\omega E_t^j[\hat{q}^{\star}_{j(t+1)} + \hat{p}_{t+1}]$$

or

(14) $$\hat{q}^{\star}_{jt} = (1-\beta\omega)\hat{\varphi}_t + \beta\omega E_t^j[\hat{q}^{\star}_{j(t+1)} + \hat{\pi}_{t+1}] \quad \Rightarrow \quad \int_0^1 \hat{q}^{\star}_{jt} = (1-\beta\omega)\hat{\varphi}_t + (\beta\omega)\int_0^1 E_t^j[\hat{q}^{\star}_{j(t+1)} + \hat{\pi}_{t+1}]\,dj.$$

Introduce the notation:

(14a) $$\hat{q}_t = \int_0^1 \hat{q}^{\star}_{jt} \quad , \quad \Phi_t(\hat{q}) = \int_0^1 \left(E_t^j\hat{q}^{\star}_{j(t+1)} - E_t^j\hat{q}_{(t+1)}\right)dj.$$

$\Phi_t(\hat{q})$ is analogous to $\Phi_t(\hat{c})$ in (6a) and both are not aggregate variables. Using (14a) I have

(15) $$\hat{q}_t = (1-\beta\omega)\hat{\varphi}_t + (\beta\omega)\bar{E}_t(\hat{q}_{t+1} + \hat{\pi}_{t+1}) + (\beta\omega)\Phi_t(\hat{q}).$$

Now recall that $\hat{q}_t = \frac{\omega}{1-\omega}\hat{\pi}_t$ hence (15) can be written as

(15a) $$\hat{\pi}_t = \frac{(1-\beta\omega)(1-\omega)}{\omega}\hat{\varphi}_t + \beta\bar{E}_t\hat{\pi}_{t+1} + \beta(1-\omega)\Phi_t(\hat{q})$$

This last term leads to the second basic proposition

Proposition 2: The forward looking Phillips Curve in the long linearized economy depends upon the market distribution of beliefs and takes the general form

$$\hat{\pi}_t = \kappa\hat{\varphi}_t + \beta\bar{E}_t\hat{\pi}_{t+1} + \beta(1-\omega)\Phi_t(\hat{q}) \quad , \quad \kappa = \frac{(1-\beta\omega)(1-\omega)}{\omega} \quad , \quad \varphi_t = \frac{1}{\zeta_t}\frac{W_t}{P_t}.$$

Diverse beliefs are expressed via the mean operator $\bar{E}$ and the extra term $\Phi_t(\hat{q})$.

From the definition of marginal cost $\hat{\varphi}_t = -\hat{\zeta}_t + (\hat{w}_t - \hat{p}_t)$ and from the first order condition for labor (5b') I have $(\hat{w}_t - \hat{p}_t) = \eta(\hat{n}_t) + \sigma(\hat{y}_t)$. Hence $\hat{\varphi}_t = -\hat{\zeta}_t + \eta\hat{n}_t + \sigma\hat{y}_t$. But from the production function I also have that $\hat{n}_t = \hat{y}_t - \hat{\zeta}_t$ hence I finally have that

$$\hat{\varphi}_t = -\hat{\zeta}_t + \eta[\hat{y}_t - \hat{\zeta}_t] + \sigma\hat{y}_t = -(1+\eta)\hat{\zeta}_t + (\eta+\sigma)\hat{y}_t .$$

I can then rewrite the Phillips Curve as

$$\hat{\pi}_t = -\kappa(1+\eta)\hat{\zeta}_t + \kappa(\eta+\sigma)\hat{y}_t + \beta\bar{E}_t\hat{\pi}_{t+1} + \beta(1-\omega)\Phi_t(\hat{q}).$$

This is a forward looking Phillips Curve except that now average expectations are not of the representative household but rather, of the diverse beliefs in the market.

Intermediate Summary of the System

Suppose the monetary rule is $\hat{r}_t = \xi_\pi\hat{\pi}_t + \xi_y\hat{y}_t + u_t$ where u_t measures random variability in the central bank's application of the rule, reflecting bank's judgment or error in special circumstances. I then have

(16a) IS Curve $\quad \hat{y}_t = \bar{E}_t(\hat{y}_{t+1}) + \Phi_t(\hat{c}) - (\frac{1}{\sigma})[\hat{r}_t - \bar{E}_t(\hat{\pi}_{t+1})]$

(16b) Phillips curve $\quad \hat{\pi}_t = \kappa(\eta+\sigma)\hat{y}_t + \beta\bar{E}_t\hat{\pi}_{t+1} + \beta(1-\omega)\Phi_t(\hat{q}) - v_t \quad , \quad v_t \equiv \kappa(1+\eta)\hat{\zeta}_t$

(16c) Monetary rule $\quad \hat{r}_t = \xi_\pi\hat{\pi}_t + \xi_y\hat{y}_t + u_t$.

This is a New Keynesian system with three endogenous variables and two exogenous shocks: a *technology supply shock* and a *bank's random policy shock*[3], with two differences from standard models. First, the extra non-aggregate terms $(\Phi_t(\hat{c}), \Phi_t(\hat{q}))$. Second, expectations are not based on a single probability measure and the operator $\bar{E}_t$ violates iterated expectations. It is an averaging over different probabilities hence $\bar{E}_t(X_{t+1})$ is merely average date t conditional forecast of X for date t+1. Such averaging among correlated random

[3] Macro models often introduce shocks without specifying their microeconomic origin and one of these is a shock to the IS curve in (16a). Since the "policy" shock u enters only through the nominal rate, it is actually equivalent to any shock to the IS curve that does not affect the system anywhere else. Hence, my view of the policy shock is that it is a proxy for any shock which is restricted only to the IS curve. On this same point I note that typically when one introduces a shock to an economic function such as utility or production, the shock may affect multiple Euler equations. It can thus affect both equations (16a)-(16b). From a modeling perspective the advantage of u is then the fact that it is restricted only to the IS curve.

variables introduces a new economic volatility which is not present in a standard models and hence it needs to be explored. The construction of an macroeconomic model depends upon the structure of market beliefs.

4. Beliefs

For beliefs to be diverse there must be something agents do not know and on which they disagree. Here I stipulate it to be the distribution of the exogenous shocks (v_t, u_t) but other exogenous shocks could be introduced and have been used in the New Keynesian literature. Now, the true process of technology and bank's policy shocks is not known. It is a non-stationary process, subject to structural changes and regime shifts due to causes I cannot discuss here (see Kurz (2009)). Following the "Rational Belief" approach (see Kurz (1994), (1997)), agents have past data on these variables hence their empirical distribution is common knowledge. By "empirical distribution" I mean the distribution one computes from a long data series by computing relative frequencies or moments and where such computations are made without judgment or attempts to estimate the effect of transitory short term events. Computation of the empirical distribution of a stochastic process leads to the formulation of a stationary probability on sequences which is then common knowledge. It plays a crucial role in the theory developed here. I denote this stationary probability with the letter m and refer to it as the "empirical distribution" or the "empirical probability." To simplify assume that (v_t, u_t) have a Markov distribution with an empirical transitions which are Markov of the form

$$\text{(17a)} \quad v_{t+1} = \lambda_v v_t + \rho^v_{t+1} \qquad \text{(17b)} \quad u_{t+1} = \lambda_u u_t + \rho^u_{t+1} \qquad \begin{pmatrix} \rho^v_{t+1} \\ \rho^u_{t+1} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma^2_v, & 0, \\ 0, & \sigma^2_u \end{bmatrix} = \Sigma \right), \quad \text{i.i.d.}$$

The truth is that both processes are subject to shifts in structure, taking the true form

$$\text{(18a)} \quad v_{t+1} = \lambda_v v_t + \lambda^s_v s_t + \tilde{\rho}^v_{t+1} \qquad \text{(18b)} \quad u_{t+1} = \lambda_u u_t + \lambda^s_u s_t + \tilde{\rho}^u_{t+1} \qquad \begin{pmatrix} \tilde{\rho}^v_{t+1} \\ \tilde{\rho}^u_{t+1} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \tilde{\sigma}^2_v, & 0, \\ 0, & \tilde{\sigma}^2_u \end{bmatrix} \right).$$

Regime parameters s_t are unobserved hence (17a)-(17b) are time averages of (18a)-(18b). To simplify it is assumed there is only one factor hence there will be one belief parameter that pins down an agent's belief about *all state variables*. More general models have multiple factors and belief variables. My aim is to discuss a general approach to belief formation that applies to a wide family of models. In some applications I examine specific examples of models with only one exogenous shock, in which case I will assume $u_t = 0$.

4.1 Describing Belief with State Variables: Rationality and Belief Diversity Imply Dynamics

Agents may believe (17a)-(17b) are the true transitions, and some do, but typically they do not and form their own beliefs about these structural parameters. I introduce agent i's state variable *denoted by* g_t^i and used to describe i's belief. It is a perception variable which pins down his subjective transition functions of all state variables. Agent i knows g_t^i but since forecast samples are taken, he observes the *distribution* of g_t^j across j but not *specific* g_t^j of others. This entails a small measure of information asymmetry as each agent knows his own g_t^i but only the distribution of the others. But this asymmetry does not matter since I also assume "anonymity." It means agent i is small and does not assume g_t^i impacts market belief. For a proper expression of anonymity suppose for a moment the economy has finite agents with a distribution $(g_t^1, g_t^2, \dots, g_t^N)$ of individual beliefs. To impose anonymity use notation of $(Z_t^1, Z_t^2, \dots, Z_t^N)$ to describe the *market distribution of beliefs* which is observed and taken by agents as given. The condition $Z_t^i = g_t^i$ is then an equilibrium condition. At no time does an agent wish to know a belief of any other specific agent. All observe past distributions $(Z_\tau^1, Z_\tau^2, \dots, Z_\tau^N)$ for $\tau < t$.

How is g_t^i used by an agent? I use the notation (v_{t+1}^i, u_{t+1}^i) to express i's *perception* of t+1 shocks before they are observed, reflecting differing views of the future. By convention I write $(E_t^i v_{t+1}, E_t^i u_{t+1})$[4] to be the same as $(E_t^i v_{t+1}^i, E_t^i u_{t+1}^i)$ since individual expectation can be taken only with respect to perception. Individual perception specifies the *difference* between date t forecast and the forecasts under the empirical probability m. Agent i's date t *perceived* distribution of (v_{t+1}^i, u_{t+1}^i) is specified to be

$$\text{(19a)} \quad v_{t+1}^i = \lambda_v v_t + \lambda_v^g g_t^i + \rho_{t+1}^{iv}$$

$$\text{(19b)} \quad u_{t+1}^i = \lambda_u u_t + \lambda_u^g g_t^i + \rho_{t+1}^{iu}$$

$$\begin{pmatrix} \rho_{t+1}^{iv} \\ \rho_{t+1}^{iu} \end{pmatrix} \sim \dot{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \sigma_{vg}^2, & \sigma_{vu}, \\ \sigma_{vu}, & \sigma_{ug}^2 \end{bmatrix} \right)$$

The assumption that $(\sigma_{vg}^2, \sigma_{ug}^2)$ is the same for all agents is made for simplicity. It follows that given public information I_t at date t, g_t^i measures the difference

$$\text{(20)} \quad E^i[(v_{t+1}, u_{t+1}) | I_t, g_t^i] - E^m[(v_{t+1}, u_{t+1}) | I_t] = (\lambda_v^g g_t^i, \lambda_u^g g_t^i).$$

I adopt two rationality principles.

Rationality Principle 1: A belief cannot be a *constant transition* unless an agent believes the stationary transition (17a)-(17b) is the truth.

[4] The notation (v_{t+1}^i, u_{t+1}^i) is used to highlight *perception* of the macro variables (v_{t+1}, u_{t+1}) by agent i before they are observed. In general, for an aggregate variable x_{t+1}, there is no difference between $E_t^i x_{t+1}^i$ and $E_t^i x_{t+1}$ since i's expectations can be taken only with respect to i's perception. However, it is important to keep in mind the context. If in a discussion the aggregate variable x_{t+1} is assumed to be observed at t+1, then it cannot be perceived at that date. In short, the notation x_{t+1}^i expresses *perception* of x_{t+1} by agent i before the variable is observed and $E_t^i x_{t+1}$ expresses the expectations of x_{t+1} by i, *in accordance with his perception*. This procedure does not apply to i-specific variables such as $E_t^i \hat{c}_{t+1}^i$ which has a natural interpretation.

<u>Rationality Principle 2</u>: A belief does not deviate from (17a)-(17b) consistently and hence the belief index g_t^i must have an unconditional mean of zero.

Condition (20) shows how to measure g_t^i using forecast data since $E^m[(v_{t+1}, u_{t+1}) | I_t]$ is a standard econometric forecast employing past data by making no judgment about special circumstances on any time interval. When $g_t^i = 0$ agent i believes m is the truth. Since beliefs are about changes in society, g_t^i reflect belief about different economies. For example, in 1900 the g_t^i are about electricity and combustion engines, but in2000 they reflected beliefs about information technology.

The two rationality principles imply that if an economy has diverse beliefs and such diversity persists without opinions tending to merge, then a typical agent's belief g_t^i *must fluctuate over time*. This is the most important implication of rationality requirements: *rationality implies dynamics*. The reason is simple. Agents cannot hold constant, invariant, transitions unless they are (17a)-(17b). Since diversity persists, (17a)-(17b) are not the belief of most but since the time average of an agent's transitions must be (17a)-(17b), they must fluctuate. This relation between rationality and dynamics is central to the Rational Belief approach (e.g. Kurz (1994), (1996), (2009)). The natural next step is the treatment of belief dynamics as state variables. Since beliefs fluctuate, such time changes of transition functions may be fixed by an agent in advance for the infinite future. More typically they are random and unknown as they may depend upon assessments made, data observed and signals received in the future. Since the first two principles do not specify the dynamics of belief, the third principle addresses the issue. To keep things simple I state it and prove it only with respect to one observed exogenous shock which, as an example, is chosen here to be v_t.

Rationality Principle 3: The transition functions of g_t^i are Markov, taking a form which exhibit persistence and if $u_t = 0$ it takes the form

(21) $g_{t+1}^i = \lambda_Z g_t^i + \lambda_Z^v [v_{t+1} - \lambda_v v_t] + \rho_{t+1}^{ig}$, $\rho_{t+1}^{ig} \sim N(0, \sigma_g^2)$ where ρ_{t+1}^{ig} are correlated across i[5].

Correlation of ρ_{t+1}^{ig} over i reflects correlated beliefs across agents and this correlation is a crucial component of the theory. Analogous law of motion applies if the shock is only u_t or both u_t and v_t.

[5] Condition (21) specifies the distribution of g_t^i hence it specifies values it will take at t+1 given the *observed* values of variables on the right hand side. For this reason one does not use perception notation here. However, in other contexts an agent takes the expectation $E_t^i g_{t+1}^i$ before g_{t+1}^i is known, at which point expectations of the perception v_{t+1}^i is taken on the right.

Rationality Principle 3 says t+1 agent belief state is unknown at t but has a Markov transition. It is analogous to the concept of a "type" in games with incomplete information where an agent type is revealed only in the future. I repeatedly use the term "forecasting belief" in the sense of taking expectations of objects like (21) or its aggregate and *uncertainty of future belief state is central to this theory*. How can one justify (21) which plays such a key role in the theory? The first answer is that the data supports this specification (see Kurz and Motolese (2011)). Alternatively, I prove (21) analytically as a result of Bayesian rationality.

4.2 Deducing (21) from a Model of Bayesian Rationality[6]

In standard Bayesian inference an agent observes data generated by a stationary process with unknown *fixed* parameter. He starts with a prior on the parameter and uses Bayesian inference for retrospective updating of his belief. The term "retrospective" stresses that inference is made *after* data is observed. In real time the prior is used for forecasting future variables while learning can improve only *future* forecasts. Under the simplification that there is only one exogenous shock v_t, with a sequence of true Markov transitions of the form (18a) with $\lambda_v^s = 1$, I have $v_{t+1} - \lambda_v v_t = s_t + \varepsilon_{t+1}^v$, $\varepsilon_{t+1}^v \sim N(0, \frac{1}{v})$ and in (18a) I have $\frac{1}{v} = \sigma_{vg}^2$. From data agents know λ_v and I assume they also know v but not the "regimes" s_t. The infinite number of time varying parameters s_t express the non stationarity of the economy. Changes reflect technologies and social organizations that define each era. Since commodities change over time, s_t represent different objects hence a single commodity is a simplification.

I now suggest that the structure of changing parameters requires us to supplement the standard Bayesian inference. To explain why note that at t-1 an agent has a prior about s_{t-1} used to forecast v_t. After observing v_t he updates the prior into a sharper posterior estimate $E_t^i(s_{t-1}|v_t)$ of s_{t-1} which, as a random variable, I denote by $s_{t-1}(v_t)$. But at date t he needs to forecast v_{t+1}. For that he does not need a posterior estimate of s_{t-1} but rather, *a new prior on b_t*! Agents do not know if and when parameters change. If they knew s_t changes slowly or $s_t = s_{t-1}$ then an updated posterior of s_{t-1} is a good prior of s_t. Without knowledge, they presume $s_{t-1} \neq s_t$ is possible and seek additional information to arrive at a sharper subjective estimate of s_t. Public *qualitative* information is an important source which offers a route to such alternative estimate.

4.2.1 *Qualitative Information As A Public Signal*

Quantitative data like v_t arrive with *qualitative* information about unusual conditions under which the data was generated. For example, if v_t are profits of a firm then v_t is a number in a financial report which contains *qualitative* information about changing consumer taste, new products, technology, joint ventures, research & development etc. If v_t reflect measures of productivity then a great deal of *qualitative* information is available about technologic discoveries, new products or new processes. If v_t is growth rate of GDP much public information is available about business conditions, public policy or political environment. Qualitative information cannot, in general, be compared over time and does not constitute conventional "data." To avoid complex

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[6] The role of belief dynamics is essential in this paper and its foundations are presented in Section 4.2. However, this section is technical in nature and a first time reader who takes (21) as given can maintain continuity of the paper's development by skipping to Section 4.3 and returning to Section 4.2 after completing the explorations of monetary policy.

modeling, I simply translate Kurz's (2008) approach to qualitative information into date t qualitative *public signal* which allows an agent to form a subjective belief about s_t. Since it is based on qualitative information it is naturally open to diverse subjective assessments. More specifically, I assume at date t, in addition to data v_t, there is a public signal leading agent i to formulate an alternate prior on s_t which, as a random variable, I denote by S_t^i defined by

$$S_t^i \sim N(\Psi_t^i, \frac{1}{\gamma}).$$

I interpret Ψ_t^i as a prior subjective mean deduced from the public signal. One can say either that i "observes" Ψ_t^i and γ or that he assesses these values from a qualitative public signal and public data. The main question is how to reconcile S_t^i with the posterior $s_{t-1}(v_t)$ formulated earlier, given the data v_t. To do that I specify the updating process.

4.2.2 *A Bayesian Inference: Beliefs are Markov State Variables with Transition (21)*

Agents believe (18a) with $\lambda_v^s = 1$ is the truth with known precision ν. At t-1 (say t-1 = -1) he forecasts v_t and uses a prior about s_{t-1} described by $s_{t-1}^i \sim N(s, \frac{1}{\alpha})$. At t (here t = 0), after observing s_t (recall $\varepsilon_{t+1}^v \sim N(0, \frac{1}{\nu})$) the posterior on s_{t-1} is updated to be

$$E_t^i(s_{t-1}|v_t) = \frac{\alpha s + \nu[v_t - \lambda_v v_{t-1}]}{\alpha + \nu} \quad , \quad s_{t-1}(v_t) \sim N[E_t^i(s_{t-1}|v_t), \frac{1}{\alpha + \nu}]$$

Using the qualitative public signal, agent i makes the assessment $S_t^i \sim N(\Psi_t^i, \frac{1}{\gamma})$ independently of the random variable $s_{t-1}(v_t)$ and we have two alternative priors. The assumption made is:

Assumption 3: With subjective probability μ agent i forms date t prior belief about s_t defined by

$$s_t(v_t, \Psi_t^i) = \mu s_{t-1}(v_t) + (1 - \mu) S_t^i \qquad 0 < \mu < 1$$

More generally, if at any stage $s_t(v_{t+1}, \Psi_t^i)$ is a posterior updated only by v_{t+1}, a revised prior given the subjective assessment S_{t+1}^i is defined by

$$s_{t+1}(v_{t+1}, \Psi_{t+1}^i) = \mu s_t(v_{t+1}, \Psi_t^i) + (1 - \mu) S_{t+1}^i \quad ,$$

$$E_{t+1}^i(s_{t+1}|v_{t+1}, \Psi_{t+1}^i) = \mu E_{t+1}^i(s_t|v_{t+1}, \Psi_t^i) + (1 - \mu)\Psi_{t+1}^i \qquad 0 < \mu < 1.$$ [7]

Theorem 1: If Assumption 3 holds then for large t $\Gamma(s_t) = \text{Precision}(s_t|v_t, \Psi_t^i)$ converges to a constant Γ^* but the Bayes estimate $E_t^i(s_t|v_t, \Psi_t^i)$ fluctuates indefinitely. Let the posterior belief of i about s_t be defined by $g_t^i = E_t^i(s_t|v_t, \Psi_t^i)$. Then this index is a Markov state variable and (21) holds

$$g_{t+1}^i = \lambda_Z g_t^i + \lambda_Z^v [v_{t+1} - \lambda_v v_t] + \rho_{t+1}^{ig} \quad , \quad \rho_{t+1}^{ig} \sim N(0, \sigma_g^2)$$

with $\rho_{t+1}^{ig} = (1 - \mu)\Psi_{t+1}^i$: Assumption 3 implies (21).

Proof: See Appendix A.

The random component $\rho_{t+1}^{ig} = (1 - \mu)\Psi_{t+1}^i$ arises from random arrival of qualitative public signals subjectively interpreted by each agent. Restrictions on the parameters (λ_Z, λ_Z^v) are explored in Appendix A and in section 4.3.

[7] $E_t^i(s_t|v_t, \Psi_t^i)$ is the notation for date t *prior* belief about s_t used to forecast v_{t+1}. I then use $E_{t+1}^i(s_t|v_{t+1}, \Psi_t^i)$ for the posterior belief *about the same* s_t given the observation of v_{t+1} but without changing the estimate of Ψ_t^i. Assumption 3 uses this posterior belief as a building block to construct the prior $E_{t+1}^i(s_{t+1}|v_{t+1}, \Psi_{t+1}^i)$ about the new parameter s_{t+1}.

4.3 Modeling Diverse Beliefs: Market Belief and the Central Role of Correlation

The fact that individual beliefs fluctuate implies market belief (i.e. the distribution of g_t^{i}) may also fluctuate and uncertainty about an agent future belief imply that future market belief is also uncertain. Indeed, market belief is a crucial macro economic uncertainty which needs to be explored.

Averaging (21), denote by Z_t the mean of the cross sectional distribution of g_t^i and refer to it as " average market belief." It is observable. Due to correlation across agents' ρ_t^{ig}, the law of large numbers does not apply and the average of ρ_t^{ig} over i does not vanish. I write it in the form

$$Z_{t+1} = \lambda_Z Z_t + \lambda_Z^v [v_{t+1} - \lambda_v v_t] + \tilde{\rho}_{t+1}^Z . \tag{22}$$

The distribution of $\tilde{\rho}_{t+1}^Z$ is unknown and may vary over time. But the fact that this random term is present reveals that the dynamics of Z_t depends upon the *correlation* across agents' beliefs. Had ρ_t^{ig} in (21) been independent across i, the law of large numbers would have implied $\tilde{\rho}_t^Z = 0$ hence the correlation ensures market belief does not degenerate into a deterministic relation $Z_{t+1} = \lambda_Z Z_t + \lambda_Z^v [v_{t+1} - \lambda_v v_t]$. Since correlation is not determined by individual rationality it becomes *an important belief externality*. In sum, random individual belief translates into macro uncertainty about future market belief. This uncertainty plays a central role in the theory and correlation externality is the basis for such uncertainty.

Since Z_t are observable, market participants have data on $\{(v_t, u_t, Z_t), t = 1, 2, \ldots\}$ and know the *joint empirical distribution* of these variables. I assume this distribution is Markov and to consider one exogenous variable at a time I have two alternative empirical distributions. The first corresponds to an economy with only technology shocks. It is described by the system with $u_t = 0$ and empirical transitions

$$v_{t+1} = \lambda_v v_t + \rho_{t+1}^v \tag{23a}$$

$$Z_{t+1} = \lambda_Z Z_t + \lambda_Z^v [v_{t+1} - \lambda_v v_t] + \rho_{t+1}^Z \tag{23b}$$

$$\begin{pmatrix} \rho_{t+1}^v \\ \rho_{t+1}^Z \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma_v^2, & 0 \\ 0, & \sigma_Z^2 \end{bmatrix} \right), \quad \text{i.i.d.}$$

The second is associated with the two shocks (v_t, u_t) with a Markov empirical probability that has a transition function described by the system of equations of the form

$$v_{t+1} = \lambda_v v_t + \rho_{t+1}^v \tag{24a}$$

$$u_{t+1} = \lambda_u u_t + \rho_{t+1}^u \tag{24b}$$

$$Z_{t+1} = \lambda_Z Z_t + \lambda_Z^v [v_{t+1} - \lambda_v v_t] + \lambda_Z^u [u_{t+1} - \lambda_u u_t] + \rho_{t+1}^Z \tag{24c}$$

$$\begin{pmatrix} \rho_{t+1}^v \\ \rho_{t+1}^u \\ \rho_{t+1}^Z \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0, \\ 0 \end{matrix} \begin{bmatrix} \sigma_v^2, & 0, & 0 \\ 0, & \sigma_u^2, & 0 \\ 0, & 0, & \sigma_Z^2 \end{bmatrix} \right), \quad \text{i.i.d.}$$

This is a combination of technology and policy shocks.

An agent who does not believe (23a)-(23b) or (24a)-(24c) are the truth, formulates his own belief-model. I describe an agent's perception of a two shocks model with the state variables $(v^i_{t+1}, u^i_{t+1}, Z^i_{t+1}, g^i_{t+1})$[8]. His belief takes the general form of a subjective perception model

$$
\begin{aligned}
&(25a)\quad v^i_{t+1} = \lambda_v v_t + \lambda^g_v g_t{}^i + \rho^{iv}_{t+1}\\
&(25b)\quad u^i_{t+1} = \lambda_u u_t + \lambda^g_u g_t{}^i + \rho^{iu}_{t+1}\\
&(25c)\quad Z^i_{t+1} = \lambda_Z Z_t + \lambda^v_Z[v_{t+1} - \lambda_v v_t] + \lambda^u_Z[u_{t+1} - \lambda_u u_t] + \lambda^g_Z g_t{}^i + \rho^{iZ}_{t+1}\\
&(25d)\quad g^i_{t+1} = \lambda_Z g_t{}^i + \lambda^v_Z[v_{t+1} - \lambda_v v_t] + \lambda^u_Z[u_{t+1} - \lambda_u u_t] + \rho^{ig}_{t+1}
\end{aligned}
\qquad
\begin{pmatrix} \rho^{iv}_{t+1} \\ \rho^{iu}_{t+1} \\ \rho^{iZ}_{t+1} \\ \rho^{ig}_{t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \hat{\sigma}^2_v, & 0, & 0, & 0 \\ 0, & \hat{\sigma}^2_u, & 0, & 0 \\ 0, & 0, & \hat{\sigma}^2_Z, & \hat{\sigma}_{Zg} \\ 0, & 0, & \hat{\sigma}^2_{Zg}, & \sigma_g \end{bmatrix} \right)
$$

(25a)-(25d) show $g_t{}^i$ pins down the transition of all state variables. This ensures one state variable pins down agent i's belief about how conditions at date t+1 are expected to be different from normal, where "normal" is represented by the empirical distribution. Comparing (24a)-(23c) with (25a)-(25d) shows that

$$E^i_t[Z_{t+1}] = \lambda_Z Z_t + \lambda^v_Z \lambda^g_v g_t{}^i + \lambda^u_Z \lambda^g_u g_t{}^i + \lambda^g_Z g_t{}^i$$

$$E^m_t[Z_{t+1}] = \lambda_Z Z_t$$

hence

$$(26)\qquad E^i_t \begin{pmatrix} v_{t+1} \\ u_{t+1} \\ Z_{t+1} \end{pmatrix} - E^m_t \begin{pmatrix} v_{t+1} \\ u_{t+1} \\ Z_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda^g_v \\ \lambda^g_u \\ \lambda^v_Z \lambda^g_v + \lambda^u_Z \lambda^g_u + \lambda^g_Z \end{pmatrix} g_t{}^i.$$

4.4 *Some A-Priori Parameter Restrictions*

The Rational Belief principle (see Kurz (1994)) restricts parameters of perception models by requiring the agent's belief, viewed as a dynamical system, to reproduce the empirical distribution which corresponds to the perception model. To illustrate consider the RBC perception model in (25a)-(25c) relative to the empirical distribution in (23a)-(23b). It can be shown that, given the unconditional variance in (21), among the restrictions imposed by the Rational Belief principle are

$$
(27)\qquad
\begin{aligned}
\mathrm{Var}[\lambda^g_v g_t{}^i + \rho^{iv}_{t+1}] &= \mathrm{Var}[\rho^v_{t+1}] \quad\Rightarrow\quad (\lambda^g_v)^2 \mathrm{Var}(g) + \hat{\sigma}^2_v = \sigma^2_v\\
\mathrm{Var}[\lambda^g_u g_t{}^i + \rho^{iu}_{t+1}] &= \mathrm{Var}[\rho^u_{t+1}] \quad\Rightarrow\quad (\lambda^g_u)^2 \mathrm{Var}(g) + \hat{\sigma}^2_u = \sigma^2_u\\
\mathrm{Var}[\lambda^g_Z g_t{}^i + \rho^{iZ}_{t+1}] &= \mathrm{Var}[\rho^Z_{t+1}] \quad\Rightarrow\quad (\lambda^g_Z)^2 \mathrm{Var}(g) + \hat{\sigma}^2_Z = \sigma^2_Z.
\end{aligned}
$$

Hence, a weak version of this principle motivates rationality restrictions such as

[8] Recall that the notation $(v^i_{t+1}, u^i_{t+1}, Z^i_{t+1})$ indicates agent i's *perception* of $(v_{t+1}, u_{t+1}, Z_{t+1})$. Since there is no difference between $E^i_t v^i_{t+1}$ and $E^i_t v_{t+1}$, I write $E^i_t v_{t+1}$ to express expectations of v_{t+1} by i, *in accordance with his perception*.

(28a) $\hat{\sigma}_v^2 < \sigma_v^2 \quad , \quad \hat{\sigma}_u^2 < \sigma_u^2 \quad , \quad \hat{\sigma}_Z^2 < \sigma_Z^2 .$

In addition, it can be shown that the variance of ρ_{t+1}^Z is restricted by σ_g^2 and is specified as

(28b) $\sigma_Z^2 < \sigma_g^2 .$

The unconditional variance of g_t^i can be calculated from the empirical distribution to be

$$\text{Var}[g] = \frac{1}{(1-\lambda_Z^2)}[(\lambda_Z^v)^2\sigma_v^2 + (\lambda_Z^u)^2\sigma_u^2 + \sigma_g^2] .$$

Without a learning feed-back this variance is only $\text{Var}[g] = \sigma_g^2/(1 - \lambda_Z^2)$. Hence, Bayesian learning feed back causes g_t^i to exhibit increased variance. Moreover, comparing the empirical distribution (24a)-(24b) with the perception model (25a)- (25b) one notes the learning feed back inevitably causes the belief variable g_t^i to introduce into (25a)- (25b) correlation with observed data which does not exist in the empirical distribution (24a)-(24b). For example, the empirical distribution shows that in the long run $\text{Cov}(v_{t+1}, v_t) = \lambda_v \text{Var}(v)$. This relation is not preserved in (25a) due to learning feed back, as can be seen in (25d). This phenomenon is general: a real time learning feed-back introduces into a subjective model relations which are absent from the data and in much of the learning literature this feed back is one of the components of volatility. A rational agent who learns in real time recognizes that his perceived model exhibits higher variance than the empirical distribution (24a)-(24c). What is a reasonable increase of variance due to a learning feed back? In most learning literature this increased variance is unrestricted hence these are actually models of bounded rationality. In the models of this paper this phenomenon is expressed by the fact that

$$(\lambda_v^g)^2 \text{Var}_g^2 + \hat{\sigma}_v^2 > \sigma_v^2 \quad , \qquad (\lambda_u^g)^2 \text{Var}_g^2 + \hat{\sigma}_u^2 > \sigma_u^2 \quad , \qquad (\lambda_Z^g)^2 \text{Var}_g^2 + \hat{\sigma}_Z^2 > \sigma_Z^2 .$$

Hence, an important issue to consider is how "modest" can the increased variance be. With a single exogenous variable such as the technology shock I set the normalization $\lambda_v^g = 1$ and measure the increased variance by the difference $(\text{Var}_g^2 + \hat{\sigma}_v^2) - \sigma_v^2$. Appendix A shows that an important way to place a-priori restrictions on the learning feed back is to deduce them from the theory itself or from empirical evidence deduced from forecast or market data. Appendix A shows $\lambda_Z^v = (\mu\nu)/(\Gamma^* + \nu)$ where ν is precision of the prior and Γ^* is limit precision of the posterior. Normally this parameter is small, perhaps 0.05-0.15. The same applies to λ_Z^u. As to λ_Z, the empirical evidence reveals (see Kurz and Motolese (2011) high persistence of mean market belief with λ_Z estimated in the range [0.6 , 0.8].

Another restriction that does not follow from the three Rationality Principle is related to dynamic stability of the perception models. Note first that (25a)-(25c) imply that

$$E_t^i[Z_{t+1}] = \lambda_Z Z_t + \lambda_Z^v g_t^i + \lambda_Z^u \lambda_u^g g_t^i + \lambda_Z^g g_t^i .$$

Hence, agents' beliefs imply

$$\int_0^1 E_t^i[Z_{t+1}]\,di = (\lambda_Z + \lambda_Z^v + \lambda_Z^u\lambda_u^g + \lambda_Z^g)Z_t.$$

As explained in Kurz (2008), dynamic stability of asset pricing requires the aggregate to exhibit stability of mean market belief. Hence, for perception models to be stable they must satisfy a condition like

(28c) $$0 \le |\lambda_Z + \lambda_Z^v + \lambda_Z^u\lambda_u^g + \lambda_Z^g| < 1.$$

With $\lambda_Z \simeq 0.6$, $\lambda_Z^v \simeq 0.15$, $\lambda_Z^u \simeq 0.10$, $\lambda_u^g = 1$ condition (28c) restricts $\lambda_Z^g > 0$.

4.5 *Definition of Equilibrium*

Having defined the belief of the agents it is useful to specify what an equilibrium in the log linearized economy entails.

Definition 2: Given a rule $\hat{r}_t = \zeta_\pi\hat{\pi}_t + \zeta_y\hat{y}_t + u_t$, an equilibrium in the log-linearized economy with two exogenous shocks is a stochastic process $\{(\hat{r}_t, \hat{p}_t, \hat{\pi}_t, \hat{w}_t, \hat{y}_t),\ t = 1, 2, ...\}$ and a collection of decision functions $(\hat{c}_t^j, \hat{b}_t^j, \hat{\ell}_t^j, \hat{n}_t^j, \hat{y}_t^j, \hat{q}_{jt}^*)$ such that

(i) decision functions are optimal for all j given j's belief (25a)-(25d),

(ii) markets clear: $\int_0^1 \hat{c}_t^j dj = \hat{y}_t$, $\int_0^1 \hat{b}_t^j dj = 0$ and $\int_0^1 \hat{\ell}_t^j dj = \int_0^1 \hat{n}_t^j dj$,

(iii) j's borrowing is bounded, transversality conditions satisfied and equilibrium is determinate.

Optimal decision functions $(\hat{c}_t^j, \hat{b}_t^j, \hat{\ell}_t^j, \hat{n}_t^j, \hat{y}_t^j, \hat{q}_{jt}^*)$ are linear in state variables but the issue at hand are the relevant state variables. An equilibrium is said to be **regular** if it is expressed with finite state variables, a finite number of lagged endogenous variables and hence it is of finite memory. If x_t is a finite vector of state variables in a regular equilibrium of the log linearized economy then, *as an equilibrium condition*, an endogenous variable w_t has a reduced form $A_w x_t$ where A_w is a vector of parameters. An equilibrium is **irregular** if it is not regular. In such equilibria endogenous variables depend on an infinite number of lagged variables or on expectation over infinite number of forward looking variables which cannot be reduced to a finite set of past or present variables. Irregular equilibria are important but analytically more difficult to simulate. This is of particular importance when we vary the monetary rule and consider later other rules which are different from $\hat{r}_t = \xi_\pi\hat{\pi}_t + \xi_y\hat{y}_t + u_t$.

One uses standard dynamic programming to show that for the economy at hand equilibria leading to (16a)-(16c) with beliefs (25a)-(25d) are **regular**. Individual decisions are functions of the state variables $(Z_t, v_t, u_t, \hat{b}_{t-1}^j, g_t^j)$ while the equilibrium map of the macro variables $(\hat{r}_t, \hat{p}_t, \hat{\pi}_t, \hat{w}_t, \hat{y}_t)$ is stated, as a set of functions of the state variables (Z_t, v_t, u_t). The difference between state spaces relevant to *each individual agent* and state spaces relevant *to the macro economy* is an important outcome of individual belief diversity.

This paper shows aggregation of equilibrium quantities is possible in the log linearized economy and hence I can construct a consistent macroeconomic model entailing structural relationships among endogenous variables. The paper also studies the impact of diverse beliefs on the performance of this long-linearized macro economy. However, it is important to clarify the relationship between the micro economic equilibrium of the long linearized economy and the macroeconomic model implied by it. To understand why recall that in a representative agent economy a macroeconomic model is a solution of dynamic optimization. Hence, a solution of the log linearized dynamic optimization is equivalent to the macroeconomic equilibrium in the log linearized economy. With diverse beliefs this is not true. I will explained below that to define the macroeconomic model one must solve the log-linearized micro economic equilibrium from which to deduce key parameters needed for the macro model. Hence, changes in policy require a reconstruction of the macro economy and to that end one must re-solve the log-linearized micro equilibrium. In short, equilibrium of the log-linearized micro economy remains a basic tool needed for the functioning of the macro model.

5. Equilibrium of the Log-Linearized Economy and the Effects of Diverse Beliefs

5.1 The Central Aggregation Result

A macro model requires a solution of the problems arising from the terms $(\Phi_t(\hat{c}), \Phi_t(\hat{q}))$ and the mean forecast operator $\bar{E}_t = \int_0^1 E_t^j dj$. The following provides a general answer to these questions.

<u>Theorem 2:</u> In an equilibrium of the log linearized economy with the policy rule $\hat{r}_t = \zeta_\pi \hat{\pi}_t + \zeta_y \hat{y}_t + u_t$

(i) there exist parameters $(\lambda_c^\Phi, \lambda_q^\Phi)$ such that $\Phi_t(\hat{c}) = \lambda_c^\Phi Z_t$ and $\Phi_t(\hat{q}) = \lambda_q^\Phi Z_t$

(ii) there exist parameters (Γ^y, Γ^π) such that

$$\bar{E}_t(\hat{y}_{t+1}) = E_t^m(\hat{y}_{t+1}) + \Gamma^y Z_t$$
$$\bar{E}_t(\hat{\pi}_{t+1}) = E_t^m(\hat{\pi}_{t+1}) + \Gamma^\pi Z_t.$$

Theorem 2 formulates transformations which are parts of the equilibrium conditions. These transformation of expectations do not hold for non-linear functions of macro variables.

<u>Sketch of a proof:</u> To explain the four parameters $(\lambda_c^\Phi, \lambda_q^\Phi, \Gamma^y, \Gamma^\pi)$ I sketch the proof of Theorem 2 for the case of two exogenous variables. Solutions of endogenous variables in the log linear economy are linear in the appropriate state variables. Keeping in mind (25a)-(25d), write the individual decision functions as

(29a) $$\hat{c}_t^{\,j} = A_y^{\,Z} Z_t + A_y^{\,v} v_t + A_y^{\,u} u_t + A_y^{\,b} \hat{b}_{t-1}^{\,j} + A_y^{\,g} g_t^{\,j} \equiv A_y \bullet (Z_t, v_t, u_t, \hat{b}_{t-1}^{\,j}, g_t^{\,j})$$

(29b) $$\hat{q}_{jt}^{\star} = \frac{\omega}{1-\omega}[A_\pi^{\,Z} Z_t + A_\pi^{\,v} v_t + A_\pi^{\,u} u_t + A_\pi^{\,b} \hat{b}_{t-1}^{\,j} + A_\pi^{\,g} g_t^{\,j}] \equiv \frac{\omega}{1-\omega} A_\pi \bullet (Z_t, v_t, u_t, \hat{b}_{t-1}^{\,j}, g_t^{\,j})$$

(29c) $$\hat{b}_t^{\,j} = A_b^{\,Z} Z_t + A_b^{\,v} v_t + A_b^{\,u} u_t + A_b^{\,b} \hat{b}_{t-1}^{\,j} + A_b^{\,g} g_t^{\,j} \equiv A_b \bullet (Z_t, v_t, u_t, \hat{b}_{t-1}^{\,j}, g_t^{\,j}).$$

Equilibrium conditions $\int_0^1 \hat{c}_t^{\,j} dj = \hat{y}_t$, $\int_0^1 \hat{b}_t^{\,j} dj = 0$, $Z_t = \int_0^1 g_t^{\,j} dj$ and $\int_0^1 \hat{q}_{jt}^{\star} = \frac{\omega}{1-\omega}\pi_t$ imply the aggregates

(29d) $$\hat{y}_t = A_y^{\,Z} Z_t + A_y^{\,v} v_t + A_y^{\,u} u_t + A_y^{\,b} 0 + A_y^{\,g} Z_t \equiv A_y \bullet (Z_t, v_t, u_t, 0, Z_t)$$

(29e) $$\hat{\pi}_t = A_\pi^{\,Z} Z_t + A_\pi^{\,v} v_t + A_\pi^{\,u} u_t + A_\pi^{\,b} 0 + A_\pi^{\,g} Z_t \equiv A_\pi \bullet (Z_t, v_t, u_t, 0, Z_t)$$

(29f) $$\hat{q}_t = \frac{\omega}{1-\omega}[A_\pi^{\,Z} Z_t + A_\pi^{\,v} v_t + A_\pi^{\,u} u_t + A_\pi^{\,b} 0 + A_\pi^{\,g} Z_t] \equiv \frac{\omega}{1-\omega} A_\pi \bullet (Z_t, v_t, u_t, 0, Z_t)$$

To compute $\Phi_t(\hat{c})$ note $\hat{c}_t = \hat{y}_t$ and use (29a)-(29f), (25a)-(25d) to deduce that

$$\int_0^1 \Big(E_t^{\,j}(\hat{c}_{t+1}^{\,j}) - E_t^{\,j}(\hat{c}_{t+1})\Big) dj = A_y^{\,g}[\int_0^1 \Big(E_t^{\,j}(g_{t+1}^{\,j}) - E_t^{\,j}(Z_{t+1})\Big) dj = -A_y^{\,g}\lambda_Z^g Z_t \text{ , hence } \lambda_c^{\Phi} = -A_y^{\,g}\lambda_Z^g.$$

$$\int_0^1 \Big(E_t^{\,j}\hat{q}_{j(t+1)} - E_t^{\,j}\hat{q}_{(t+1)}\Big) dj = -\frac{\omega}{1-\omega} A_\pi^{\,g}\lambda_Z^g Z_t \text{ , hence } \lambda_q^{\Phi} = -\frac{\omega}{1-\omega} A_\pi^{\,g}\lambda_Z^g.$$

Using the same information and (26), compute now the expression

$$\int_0^1 [E_t^{\,j}\hat{y}_{t+1} - E_t^{\,m}\hat{y}_{t+1}] = (A_y^{\,Z} + A_y^{\,g})\int_0^1 [E_t^{\,j}Z_{t+1} - E_t^{\,m}Z_{t+1}] + A_y^{\,v}\int_0^1 [E_t^{\,j}v_{t+1} - E_t^{\,m}v_{t+1}] + A_y^{\,u}\int_0^1 [E_t^{\,j}u_{t+1} - E_t^{\,m}u_{t+1}]$$

$$= \Big((A_y^{\,Z} + A_y^{\,g})[\lambda_Z^v\lambda_v^g + \lambda_Z^u\lambda_u^g + \lambda_Z^g] + A_y^{\,v}\lambda_v^g + A_y^{\,u}\lambda_u^g\Big) Z_t$$

Similar argument holds with respect to inflation hence I have that

(29g) $$\Gamma^y = (A_y^{\,Z} + A_y^{\,g})[\lambda_Z^v\lambda_v^g + \lambda_Z^u\lambda_u^g + \lambda_Z^g] + A_y^{\,v}\lambda_v^g + A_y^{\,u}\lambda_u^g$$

(29h) $$\Gamma^\pi = (A_\pi^{\,Z} + A_\pi^{\,g})[\lambda_Z^v\lambda_v^g + \lambda_Z^u\lambda_u^g + \lambda_Z^g] + A_\pi^{\,v}\lambda_v^g + A_\pi^{\,u}\lambda_u^g.$$

Note these transformations do not hold for, say $(\hat{y}_t)^2$ which is not a linear function of state variables. ■

To study the system I transform it into one in which the expectation operator obeys the law of iterated expectations so as to enable us to use standard techniques of analysis. To do that I observe that for defining the macro model one needs only the two parameters defined by

$$B_y = \lambda_c^{\Phi} + \Gamma^y + (\frac{1}{\sigma})\Gamma^{\pi} \quad , \qquad\qquad B_{\pi} = (1-\omega)\lambda_q^{\Phi} + \Gamma^{\pi} \quad .$$

Using the theorem above I can now rewrite the system (16a) - (16c) in the form

(30a) IS Curve $\hat{y}_t = E_t^m(\hat{y}_{t+1}) + B_y Z_t - (\frac{1}{\sigma})[r_t - E_t^m(\hat{\pi}_{t+1})]$

(30b) Phillips curve $\hat{\pi}_t = \kappa(\eta+\sigma)\hat{y}_t + \beta E_t^m \hat{\pi}_{t+1} + \beta B_{\pi} Z_t - v_t \quad , \quad v_t \equiv \kappa(1+\eta)\hat{\zeta}_t$

(30c) Monetary rule $\hat{r}_t = \xi_{\pi}\hat{\pi}_t + \xi_y \hat{y}_t + u_t$

together with the law of motion of (v_t, u_t, Z_t) under the empirical transitions (24a)-(24c). Since this system is operative under a single probability law m which satisfies the law of iterated expectations, standard methods of Blanchard Kahn (1980) are applicable for setting conditions to ensure determinacy.

The system at hand shows that diverse beliefs have two effects. First, the mean market belief Z_t has an amplification effect on the dynamics of the economy. The second is more subtle. To explain it note the probability in (30a)-(30b) is m, not the true dynamics (18a)-(18b) that is unknown to anyone and simulations are conducted with respect to the empirical probability m. Hence, (30a)-(30c) may not reflect big changes in s_t (see (18a)-(18b)) if they are not predicted by the public and expressed in Z_t. But this fact shows that a central bank faces an enduring problem for which only imperfect solution exist. To capture what it does not observe, the bank has two options. One is to base policy upon market belief, expressed either directly by Z_t or by asset prices which are functions of Z_t (see Kurz and Motolese (2011)). A second option is to use the bank's own belief model in making policy decisions, giving rise to what the market would view as random policy shocks, which may turn out to be costly in becoming an independent cause of volatility. I return to this subject later when I discuss the implications to monetary policy.

5.2 *Some Characteristics of the Micro Economic Equilibrium*

It follows from (29g)-(29h) that (B_y, B_{π}) are functions of (A_y, A_{π}, A_b) which is an equilibrium of the log-linearized micro economy. That is, to deduce a solution of the macro model (30a)-(30c), one must first obtain a *micro equilibrium* solution of (A_y, A_{π}, A_b). Note that an equilibrium depends upon the model parameters including policy parameters. Since we study the effect of different policy parameter, the shape of the map from parameters to equilibria (A_y, A_{π}, A_b) is important. For this reason I use the term Equilibrium Manifold to describe the set of equilibria (A_y, A_{π}, A_b) as a function of the model's parameters.

Appendix B reviews computation of (A_y, A_{π}, A_b) for a simple model of a technology shock with $u_t = 0$. Exploring this model further, note it is a system with endogenous variables $(\hat{\pi}_t, \hat{y}_t)$ and shocks (v_t, Z_t). The primary uncertainty are technology shocks. Rewriting this complete aggregate system we have

$$\text{(31a)}\quad v_{t+1} = \lambda_v v_t + \rho^v_{t+1} \qquad \begin{pmatrix} \rho^v_{t+1} \\ \rho^Z_{t+1} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma_v^2, & \sigma_{vZ} \\ \sigma_{vZ}, & \sigma_Z^2 \end{bmatrix}\right)$$

$$\text{(31b)}\quad Z_{t+1} = \lambda_Z Z_t + \lambda^v_Z[v_{t+1} - \lambda_v v_t] + \rho^Z_{t+1}$$

$$\text{(31c)}\quad E_t^m(\hat{y}_{t+1}) = \hat{y}_t + (\frac{1}{\sigma})[\xi_\pi \hat{\pi}_t + \xi_y \hat{y}_t - E_t^m(\hat{\pi}_{t+1})] - B_y Z_t \quad , \qquad B_y = \lambda_c^\Phi + \Gamma^y + (\frac{1}{\sigma})\Gamma^\pi$$

$$\text{(31d)}\quad E_t^m \hat{\pi}_{t+1} = \frac{1}{\beta}\hat{\pi}_t - \frac{\kappa(\eta+\sigma)}{\beta}\hat{y}_t - B_\pi Z_t - \frac{1}{\beta}v_t \quad , \qquad B_\pi = (1-\omega)\lambda_q^\Phi + \Gamma^\pi \ , \ \kappa = \frac{(1-\beta\omega)(1-\omega)}{\omega}.$$

(30a)-(30c) together with (31a)-(31d) show that endogenous variables do not affect the dynamics of either the exogenous shock or the dynamics of belief. It then follows that we have:

Proposition 3: Determinacy of equilibrium is not affected by diversity of beliefs.

Proofs: It follows from Blanchard-Kahn (1980) that to compute the relevant eigenvalues one ignores the first two equations. For the case $u_t = 0$, the condition for determinacy when $\xi_y \geq 0$, $\xi_\pi \geq 0$ is

$$\text{(32)}\qquad \xi_y(1-\beta) + (\xi_\pi - 1)\kappa(\eta+\sigma) > 0.$$

It does not involve belief parameters and is the same as an equivalent model with homogenous beliefs. ■

Does Proposition 3 mean that existence and uniqueness are the same as they would be without diverse beliefs? The answer is No. To explain why, I continue to study the simple version of the model with a single exogenous technology shock. I explore now some features of the equilibrium system.

Proposition 4: It is impossible to solve the macro model using only the aggregate system (31a)-(31d). To solve (31a)-(31d) one must first deduce (B_y, B_π) from a micro equilibrium of the log-linearized economy underlying (31a)-(31d).

Proof: It is explained in Appendix B that equilibrium values (A_y, A_π, A_b) of the micro model are deduced from the log-linearized Euler equations (5a) and (14) which I write in the form

$$\hat{c}_t^j + (\frac{1}{\sigma})[\zeta_\pi \pi_t + \zeta_y \hat{y}_t] = E_t^j(\hat{c}_{t+1}^j) + (\frac{1}{\sigma})E_t^j(\pi_{t+1}) + \tau_b \hat{b}_t^j$$

$$\frac{1-\omega}{\omega}\hat{q}^\star_{jt} = -v_t + \kappa(\eta+\sigma)\hat{y}_t + \beta(1-\omega)E_t^j[\hat{q}^\star_{j(t+1)} + \pi_{t+1}].$$

By (29a)-(29f) one writes these equations in the following linear form in j's expected values

$$\text{(33a)}\quad A_y\bullet(Z_t, v_t, \hat{b}^j_{t-1}, g^j_t) + \frac{\zeta_\pi}{\sigma}[A_\pi\bullet(Z_t, v_t, 0, Z_t)] + \frac{\zeta_y}{\sigma}[A_y\bullet(Z_t, v_t, 0, Z_t)] =$$

$$= A_y\bullet\left(E_t^j[Z_{t+1}], E_t^j[v_{t+1}], \hat{b}_t^j, E_t^j[g^j_{t+1}]\right) + (\frac{1}{\sigma})A_\pi\bullet\left(E_t^j[Z_{t+1}], E_t^j[v_{t+1}], 0, E_t^j[Z_{t+1}]\right) + \tau_b A_b\bullet(Z_t, v_t, \hat{b}^j_{t-1}, g^j_t)$$

(33b) $A_\pi \bullet (Z_t, v_t, \hat{b}_{t-1}^j, g_t^j) - \kappa(\eta+\sigma) A_y \bullet (Z_t, v_t, 0, Z_t) =$

$$= -v_t + \beta\omega A_\pi \bullet \left(E_t^j[Z_{t+1}], E_t^j[v_{t+1}], \hat{b}_t^j, E_t^j[g_{t+1}^j]\right) + \beta(1-\omega) A_\pi \bullet \left(E_t^j[Z_{t+1}], E_t^j[v_{t+1}], 0, E_t^j[Z_{t+1}]\right)$$

and expectations defined by

$$E_t^j[v_{t+1}^j] = \lambda_v v_t + \lambda_v^g g_t^j$$
$$E_t^j[Z_{t+1}] = \lambda_Z Z_t + \lambda_Z^v g_t^j + \lambda_Z^g g_t^j$$
$$E_t^j[g_{t+1}^j] = \lambda_Z g_t^j + \lambda_Z^v g_t^j .$$

The 12 equilibrium function values (A_y, A_π, A_b) are determined by matching coefficients of the four state variables across equations. Inserting the expectation values into (33a)-(33b) one obtains 8 equations in the 12 unknown equilibrium values. It is shown in Appendix B that the final four restrictions follow from the optimal borrowing function which is deduced from the budget constraint of agent j. Under the insurance assumption, this equation is defined by

(33c) $$\hat{b}_t^j = \frac{1}{\beta}\hat{b}_{t-1}^j + [1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}](\hat{y}_t - \hat{c}_t^j).$$

Return now to (29a) - (29d) and by matching coefficients one deduces that the final four restrictions are

(33d) $A_b^Z = A_y^g[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}]$, $A_b^v = 0$, $A_b^b = \frac{1}{\beta} - A_y^b[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}]$, $A_b^g = -A_y^g[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}]$.

This procedure cannot be carried out for (31a)-(31d). Using (29d)-(29e) one can write it in a linear form, and even by using the borrowing restriction (33c) one can deduce only eight equations in 12 unknowns. ■

Proposition 4 raises questions of existence and uniqueness of equilibria (A_y, A_π, A_b) in the log linearized economy. Note that the system of equations implied by parameter matching of (33a)-(33c) is non-linear due to the presence of *products* which arise from the borrowing function. Indeed, there are 8 products: $(A_y^b A_b^Z, A_y^b A_b^v, A_y^b A_b^v, A_y^b A_b^g)$ in (33a) and $(A_\pi^b A_b^Z, A_\pi^b A_b^v, A_\pi^b A_b^b, A_\pi^b A_b^g)$ in (33b). A close inspection leads to several observations.

Proposition 5: The equation system defining equilibrium for $\tau_b > 0$ is non linear with at least two solutions where one entails explosive optimal borrowing, independent of determinacy conditions.

Proof: Matching parameters of the state variable $\hat{b}_{t-1}^j$ in (33a)-(33b) leads to two non-linear equations

(34a) $$A_y^b(1 - A_b^b) = \tau_b A_b^b$$

(34b) $$A_\pi^b = \beta\omega A_\pi^b A_b^b.$$

Now, $A_y^b = 0 \Rightarrow A_b^b = 0$ due to (34a) and $A_b^b = \frac{1}{\beta} > 0$ due to (33d). This is a contradiction, hence $A_y^b \neq 0$. (33d) and (34a) imply that

(34c) $$A_b^b = \frac{A_y^b}{A_y^b + \tau_b} = \frac{1}{\beta} - A_y^b[1 + \frac{\theta - 1}{\theta}\frac{\sigma}{\eta}] \quad \text{hence} \quad A_y^b = [\frac{1}{\beta} - \frac{A_y^b}{A_y^b + \tau_b}]\frac{1}{[1 + \frac{\theta - 1}{\theta}\frac{\sigma}{\eta}]}$$

If $A_\pi^b \neq 0$ it follows that $A_b^b = \frac{1}{\beta\omega}$ which contradicts (34c) hence $A_\pi^b = 0$. This implies that

$$A_\pi^b A_b^Z = 0 \quad , \quad A_\pi^b A_b^v = 0 \quad , \quad A_\pi^b A_b^g = 0 .$$

Now use (33d) and (34c) to deduce

$$A_y^b A_b^Z = [\frac{1}{\beta} - \frac{A_y^b}{A_y^b + \tau_b}]A_y^g$$

$$A_y^b A_b^v = 0$$

$$A_y^b A_b^g = -[\frac{1}{\beta} - \frac{A_y^b}{A_y^b + \tau_b}]A_y^g .$$

Next, let $\Xi = 1 + \frac{\theta - 1}{\theta}\frac{\sigma}{\eta}$ which is positive since $\theta > 1$. Then, (34c) implies the equation

$$(A_y^b)^2 - \frac{1}{\Xi}(\frac{1-\beta}{\beta} - \tau_b)A_y^b - \frac{\tau_b}{\beta\Xi} = 0$$

for which there are two *exact* solutions

(35) $$A_y^b = \frac{\frac{1}{\Xi}(\frac{1-\beta}{\beta} - \tau_b) \pm \sqrt{\frac{1}{\Xi^2}(\frac{1-\beta}{\beta} - \tau_b)^2 + 4\frac{\tau_b}{\beta\Xi}}}{2}$$

one positive and one negative. Indeed, these are approximately

$$A_y^b \cong \frac{1}{\Xi}(\frac{1-\beta}{\beta}) > 0 \qquad , \qquad A_y^b \cong -\frac{\tau_b}{|\sqrt{\beta\Xi}|} < 0 .$$

To deduce equilibrium insert a solution of (35) into the six products. (33a)-(33b) then imply six linear equations in the six parameters (A_y, A_π). But two solutions of A_y^b imply two solutions for (A_y, A_π, A_b). Since A_y^b measures the effect of bond holdings on consumption, $A_y^b < 0$ imply increased consumption and borrowing when in debt and this causes individual debt to diverge *for any* τ_b. This second solution is thus not an equilibrium! For a positive penalty the only equilibrium is the one implied by $A_y^b > 0$ in (35). ■

The dynamic determinacy condition (32) plays no role in Proposition 5. This implies that determinacy restricts the micro economic equilibrium map only in part. I thus further explore other properties of the equilibrium map (e.g. singularity, which impacts individual borrowing) in relation to determinacy.

Selecting the non-explosive solution of A_y^b and $A_\pi^b = 0$ as in Proposition 5 reduces the system to six linear equations in six unknowns (A_y, A_π). Denote this equation system by $MA = h$ and inspection of (33a)-(33b) reveals that the right hand vector h contains *only parameters of the process of exogenous shocks*. Hence, altering these shocks alters the equilibrium (A_y, A_π, A_b). I refer to the determinant $|\mathbf{M}|$ as the *"Equilibrium Determinant."* It helps understand the impact of policy on equilibrium since changes in policy parameters change (A_y, A_π, A_b) and (B_y, B_π) in (31a)-(31d), and as we sweep over the feasible space of policy parameters (ξ_y, ξ_π) the determinant changes. I state without proof the fact that

Proposition 6: As one varies policy parameters over the two dimensional space, the Equilibrium Determinant takes the value zero and changes sign, but any singularity occurs outside the region of determinacy.

Proposition 6 clarifies two issues. First, the equilibrium map of the log linearized economy does have singularities. Second, any singularity occurs outside the region of determinacy. This means that over the set of points satisfying the conditions of determinacy, changes in (ξ_y, ξ_π) have a *continuous* effect on (σ_y, σ_π). This conclusion justifies the simulation work, Tables and Graphs presented later. It is then only natural to raise two other questions which are pivotal to this paper and are central to the impact of policy on (σ_y, σ_π):

(i) Do (ξ_y, ξ_π) have monotonic effects on (σ_y, σ_π)?

(ii) What is the policy tradeoff, if any, between σ_y and σ_π, and what is the effect of diverse beliefs on such a tradeoff?

These will be the central questions studied, via simulations, in the next Section.

6. Simulation Study of the Impact of Diverse Beliefs on Feasible Monetary Policy Outcomes (Work joint with Giulia Piccillo and Howei Wu)

6.1 On Output Difference and Output Gap

Some New Keynesian models under RE use output level under flexible prices as a yardstick for central bank policy. Under these conditions $\hat{y}_t^{\,f}$, log deviation of output from steady state, is not a function of prices or expectations but only a function of the technology shock. Indeed, we can derive the relationship

$$\hat{y}_t^{\,f} = \left(\frac{1+\eta}{\sigma+\eta}\right)\hat{\zeta}_t = \frac{1}{(\sigma+\eta)\kappa} v_t .$$

Inserting this definition into the Phillips Curve (30b) transforms it into

$$\pi_t = \kappa(\eta+\sigma)[\hat{y}_t - \hat{y}_t^f] + \beta E_t^m \pi_{t+1} + \beta B_\pi Z_t \equiv \kappa(\eta+\sigma)\hat{x}_t + \beta E_t^m \pi_{t+1} + \beta B_\pi Z_t \tag{36}$$

where $\hat{x}_t = \hat{y}_t - \hat{y}_t^f$. The rest (30a)-(30c) is then redefined in terms of $\hat{x}_t$, including the monetary policy rule. Such a transformation is *equivalent* to solving the untransformed system in $(\hat{y}_t, \pi_t)$ but altering the policy rule to be $\hat{r}_t = \xi_\pi \pi_t + \xi_y [\hat{y}_t - \frac{1}{(\sigma+\eta)\kappa} v_t]$ and no "output gap" needs be defined at all. The justification for this change in rule is that competitive equilibrium under flexible prices is the *first best* and hence policy should aim to attain it. This argument fails when we have diverse beliefs and/or other shocks such as a policy shock, since then the model under flexible prices is not an RBC model and $\hat{y}_t^f$ is neither first best nor does it have any welfare significance. I noted earlier the results of Kurz, Jin and Motolese (2005) who show that diverse beliefs call, *on their own*, for stabilization policy that would counter the volatility amplification of market belief. In their paper the central objective of monetary policy is to stabilize the volatility amplification of market expectations. Under such circumstances the policy objective should be $\hat{y}_t$ itself, which reflects the effect of belief, not $\hat{x}_t$. This is the procedure I follow in this paper. One can solve (36) by iterating forward and the solution of π_t is

$$\pi_t = \sum_{\tau=0}^{\infty} \beta^\tau E_t^m [\kappa(\eta+\sigma)\hat{x}_{t+\tau} + \beta B_\pi Z_{t+\tau}] \tag{36a}$$

showing that the Blanchard & Gali (2007) "divine coincidence" does not hold. This is even more pronounced when other shocks, such as taste or cost plus shocks, are present. Nevertheless, since $Z_{t+\tau}$ are not altered by policy, it is a purely a mathematical observation based on (36a) that any anti inflation policy which reduces the volatility of $\hat{\pi}_t$ will also reduce the volatility of $\hat{x}_t$, although other factors may be considered for a better policy choice. More specifically, in Section 6.3 I test the use of a rule that targets the gap $\hat{x}_t$ vs. alternative rule which I propose for targeting the causes of output volatility instead of output.

6.2 *Monetary Policy Tradeoffs Under Rational Expectations*

Before proceeding to study diverse beliefs, it is useful to clarify the results under RE. I thus consider the results of the RBC model (31a)-(31d) with $u_t = 0$ under RE, where all believe (24a) is the truth. Table 1 reports the results for $(\xi_y \geq 0, \xi_\pi \geq 1)$ which is the standard range used in most literature on monetary policy. The model's parameters are standard (e.g. Galí (2008)) and will be maintained throughout this paper:

$$\beta = 0.98\,,\ \sigma = 0.9\,,\ \eta = 1.0\,,\ \tau_b = 10^{-4}\,,\ \omega = \frac{2}{3}\,,\ \theta = 6\,,\ \lambda_v = 0.90.$$

The standard RBC assumption of $\sigma_\zeta = 0.0072$ (measured for the "Solow Residuals") is being used and I comment on this matter later. Also, recall that the empirical record for the US exhibits $\sigma_y = 1.81$, $\sigma_\pi = 1.79$.

Table 1.1: RBC Volatility and Monotonicity under Rational Expectations

Standard RBC with $\sigma_\zeta = 0.0072$

σ_y		ξ_y 0.0	0.6	1.2	1.5	30	σ_π		ξ_y 0.0	0.6	1.2	1.5	30
ξ_π	1.0	1.31	0.50	0.31	0.26	0.02	ξ_π	1.0	1.18	3.43	4.05	4.07	4.78
	1.6	1.67	1.29	1.05	0.95	0.11		1.6	0.21	1.28	1.93	2.17	4.52
	2.2	1.69	1.46	1.28	1.21	0.19		2.2	0.12	0.77	1.27	1.48	4.31
	2.8	1.70	1.54	1.40	1.33	0.26		2.8	0.08	0.56	0.95	1.12	4.13
	30	1.75	1.74	1.72	1.72	1.27		30	0.01	0.04	0.08	0.09	1.31

Table 1.1 shows that for $\sigma_\zeta = 0.0072$ there are policy configurations for which the simulated values are within the range of the data. More important is the effect of policy. The conclusions are clear:

- Increasing ξ_y results in a *monotonic* decrease of σ_y and a monotonic increase of σ_π
- Increasing ξ_π results in a *monotonic* increase of σ_y and a monotonic decrease of σ_π.

These results imply a policy trade-off between σ_y and σ_π which entails a central bank's choice between fighting inflation and stabilizing output. Using simulated data, Figure 1.1 shows that under RE such a tradeoff exists over the region $(\xi_y \geq 0, \xi_\pi \geq 1)$. These results are consistent with the RE based studies of Taylor (1979), Fuhrer (1994), Ball (1999) and Rudebusch and Svensson (1999) . Although the steps of parameter change are wide, choice of smaller steps shows the results are continuous with respect to policy.

FIGURE 1.1 PLACE HERE

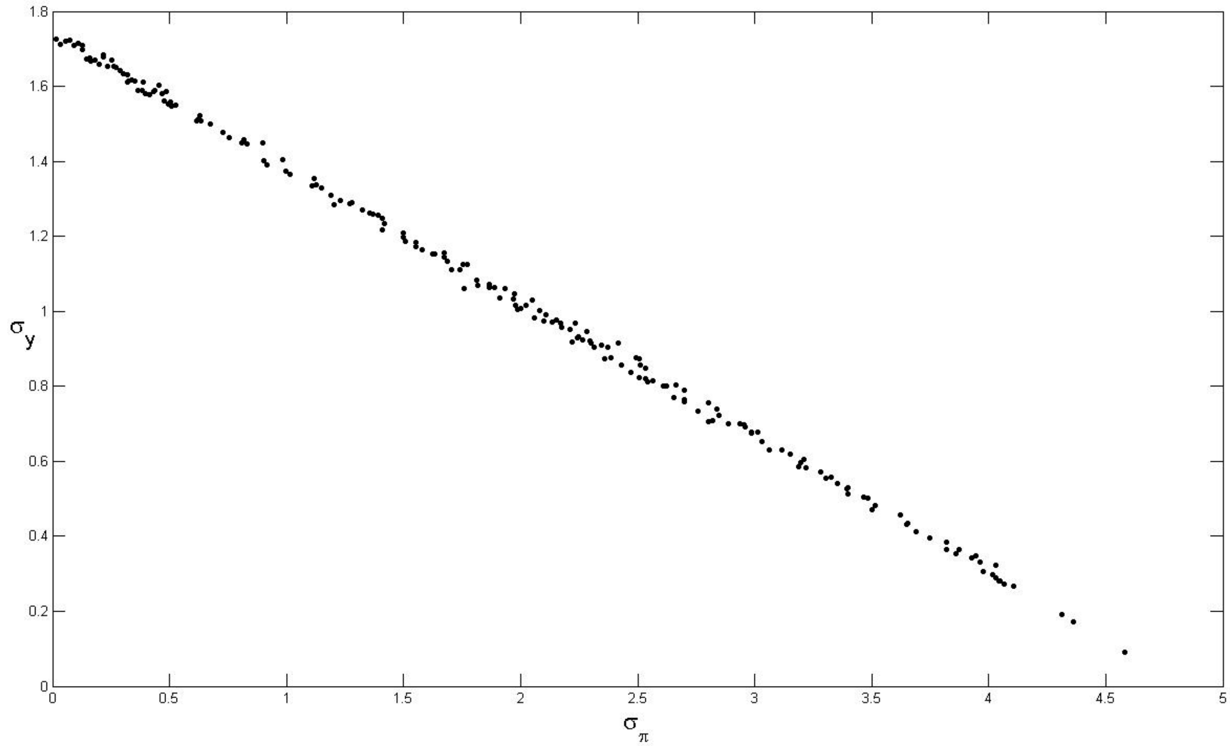
(see next page)

Why study monotonicity of response to policy parameters? The above results provide a partial answer. First, monotonic response to policy parameters is a precondition for the trade-off exhibited by the RE based model in Figure 1.1 and by the models of the above cited authors. A second reason for interest in monotonicity is that monotonic response renders the impact of policy predictable since it means the central bank knows the *direction* of the effect of increasing weight of any policy instrument. In reality no central bank knows the exact effect of its policy instruments. Hence, a policy with non-monotonic effect in the region of determinacy means that a bank is uncertain not only about the size or timing of the effect of policy but even about the *direction* of such effects. This is an undesirable position for a bank to be in. I will thus suggest that if an economy exhibits threshold regions with non-monotonic response to central bank policy actions, it is appropriate to consider alternative policy rules with effects which are monotonic in well defined regions.

Continuing to explore the response under RE, I examine now a two shock (v, u) model under the RE specification (24a)-(24b) with a wider policy space which allows $(\xi_y < 0, \xi_\pi \geq 1)$, but revise the determinacy conditions. $\xi_\pi < 1$ clearly violates determinacy. It follows from Proposition 1 of Bullard and

Figure 1.1: Frontier under the Taylor Rule Implied by Table 1.1

The grid used is for $\xi_\pi = 1.1, 1.35, 1.6, 1.85, 2.1, 10$ and $0 \leq \xi_y \leq 3$, plus the points $\xi_y = 5, 10$.



Mitra (2002) that for determinacy to hold when $\xi_y < 0$, condition (32) is supplemented by the condition

(32a) $$\xi_y + \kappa(\eta + \sigma)\xi_\pi > -(1 - \beta)\sigma .$$

Table 1.2 presents results for the *two shock* model under RE for the wider policy space. In this table the region under the bold line delineates policy parameters that satisfy both determinacy conditions (32) and (32a). The table exhibits two interesting regions: (i) (σ_y, σ_π) which are to the left of the vertical lines (one in purple and one in yellow) identified approximately by $\xi_y = 0.2$ in the σ_y space and by $\xi_y = 0.1$ in the σ_π space; (ii) the collection of all (σ_y, σ_π) to the right of these two vertical lines. These two vertical "thresholds" delineate where the *effect of policy exhibits non-monotonicity and\or reversal of direction*.

TABLE 1.2 PLACE HERE

(see next page)

In Table 1.2 the volatilities (σ_y, σ_π) exhibit the following results:

(i) σ_y exhibits monotonic *decline* in ξ_y over the entire determinacy region,

(ii) σ_y exhibits monotonic *decline* in ξ_π for $\xi_y < 0.2$ but then a monotonic *increase* in ξ_π for $\xi_y > 0.2$,

(iii) σ_π exhibits monotonic *decline* in ξ_π over the entire determinacy region,

(iv) σ_π exhibits monotonic *decline* in ξ_y for $\xi_y < 0.1$ but then a monotonic *increase* in ξ_y for $\xi_y > 0.1$.

Conditions (ii) and (iv) show the results in Table 1.2 are different from the strict monotonicity in Table 1.1. Although, as we see, the effects of ξ_π on σ_y in (ii) and ξ_y on σ_π in (iv) are non-monotonic, in practice monotonicity does hold in a narrow sense since the impact of a policy parameter on volatility reaches a threshold, and after passing it the effect is monotonic. Policy makers need only to approximate the threshold.

Table 1.2 records policy response under RE and under a second shock u with $\sigma_u = 0.006$ (estimated from the data). Hence, even under RE one finds that response to policy exhibit reversals in the direction of the impact of policy. This suggests that in an economy with multiple shocks (i.e. the normal case), regions of non- monotonic response and reversals in the effect of policy are normal patterns even under RE. To explore this further, I report the results of policy tradeoff under RE but with an artificially large shock where I set $\sigma_u = 0.018$. Figures 1.2 and 1.3 report the policy frontiers deduced from simulation of the model under RE, for the Taylor rule $\hat{r}_t = \xi_\pi \hat{\pi}_t + \xi_y \hat{y}_t + u_t$ and for the Output Gap rule $\hat{r}_t = \xi_\pi \pi_t + \xi_y [\hat{y}_t - \frac{1}{(\sigma + \eta)\kappa} v_t] + u_t$.

Figure 1.2 shows that adding a large shock in the IS curve completely alters the policy tradeoff seen in Figure 1.1 for a single technology shock. Such a shock eliminates all tradeoff between inflation and output stabilization. Both are dominated by a common factor and a stabilization policy has to fight its impact on the market. If interest rate policy can neutralize the effect of the shock, it stabilizes both output and inflation. More generally, an examination of economies with multiple shocks shows that response to policy depends

Table 1.2: (u,v) Model Volatility and Monotonicity Properties under Rational Expectations

with σ_ζ=0.0072, σ_u=0.006 ($-0.6 \leq \xi_y \leq 30$, $1.1 \leq \xi_\pi \leq 30$)

Standard deviation of Output, y

σ_y	ξ_y → -0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1	1.5	2	2.5	3	3.5	4	30
ξ_π = 1.1	n.a.	n.a.	3.15	2.16	1.66	1.34	1.12	0.96	0.85	0.64	0.52	0.44	0.38	0.34	0.30	0.04
1.3	n.a.	n.a.	2.34	1.92	1.61	1.43	1.26	1.14	1.02	0.82	0.69	0.61	0.52	0.47	0.42	0.07
1.5	n.a.	2.42	2.08	1.81	1.65	1.48	1.35	1.22	1.14	0.96	0.83	0.73	0.65	0.59	0.54	0.10
1.7	n.a.	2.18	2.00	1.78	1.64	1.52	1.40	1.31	1.24	1.06	0.93	0.85	0.76	0.70	0.64	0.12
1.9	2.26	2.07	1.92	1.77	1.65	1.54	1.46	1.36	1.30	1.16	1.01	0.93	0.85	0.79	0.73	0.15
2.1	2.15	2.01	1.86	1.75	1.64	1.56	1.49	1.42	1.34	1.21	1.09	1.01	0.92	0.86	0.81	0.18
2.6	2.00	1.89	1.84	1.75	1.66	1.61	1.54	1.49	1.42	1.31	1.23	1.13	1.05	0.99	0.94	0.24
3.1	1.91	1.82	1.79	1.74	1.69	1.62	1.57	1.54	1.49	1.40	1.30	1.24	1.15	1.11	1.05	0.30
3.6	1.86	1.82	1.79	1.74	1.70	1.65	1.60	1.57	1.53	1.46	1.39	1.28	1.23	1.19	1.14	0.35
4.1	1.82	1.80	1.80	1.74	1.68	1.64	1.63	1.60	1.55	1.48	1.40	1.37	1.30	1.24	1.20	0.40
30	1.75	1.72	1.75	1.72	1.75	1.73	1.75	1.73	1.70	1.70	1.69	1.69	1.67	1.68	1.64	1.25

Standard deviation of Inflation, π

σ_π	ξ_y → -0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1	1.5	2	2.5	3	3.5	4	30
ξ_π = 1.1	n.a.	n.a.	3.73	2.47	2.43	2.71	2.95	3.17	3.31	3.64	3.82	4.04	4.12	4.18	4.22	4.73
1.3	n.a.	n.a.	2.13	1.65	1.66	1.88	2.12	2.31	2.53	2.89	3.20	3.48	3.56	3.71	3.84	4.67
1.5	n.a.	2.10	1.49	1.25	1.27	1.45	1.65	1.82	2.03	2.41	2.71	2.98	3.15	3.31	3.48	4.57
1.7	n.a.	1.53	1.15	1.01	1.04	1.18	1.35	1.53	1.71	2.07	2.37	2.65	2.83	3.00	3.15	4.47
1.9	1.61	1.20	0.95	0.84	0.87	0.99	1.14	1.30	1.47	1.83	2.09	2.36	2.58	2.76	2.92	4.44
2.1	1.31	1.00	0.79	0.72	0.75	0.86	1.00	1.15	1.27	1.61	1.89	2.14	2.33	2.54	2.71	4.37
2.6	0.88	0.69	0.56	0.53	0.56	0.65	0.75	0.87	0.98	1.26	1.51	1.74	1.91	2.09	2.23	4.20
3.1	0.66	0.53	0.44	0.42	0.45	0.51	0.61	0.70	0.80	1.05	1.25	1.46	1.62	1.82	1.95	4.03
3.6	0.53	0.43	0.37	0.35	0.37	0.43	0.51	0.59	0.67	0.89	1.10	1.23	1.41	1.59	1.72	3.88
4.1	0.44	0.36	0.31	0.30	0.32	0.37	0.43	0.51	0.58	0.77	0.94	1.12	1.26	1.39	1.54	3.78
30	0.05	0.04	0.03	0.03	0.04	0.04	0.05	0.06	0.07	0.10	0.13	0.15	0.18	0.21	0.23	1.29

Note: In the region below the bold lines the parameters satisfy Blanchard-Kahn conditions.

upon the system's shocks. In that case the existence of thresholds, where the effect of policy reverses direction, is a universal phenomenon. *Under multiple shocks a policy tradeoff may be restricted or it may not exist.* This observation is important as it will help understand the effect of diverse beliefs on the response to policy.

The above conclusions hold with respect to policy under the Output Gap. As I pointed out in Section 6.1 the Output Gap rule has little justification for economies which are not pure RBC economies with a single technological shock. Figure 1.3 shows that in the presence of a large u shock in the IS curve all tradeoff between output and inflation is eliminated.

FIGURES 1.2-1.3 PLACE HERE

(see next page)

6.3 Impact of Diverse Beliefs in the (v, u) Model and the Problem of Individual Consumption Volatility

Before presenting the results for economies with diverse beliefs it is useful to provide some intuition on how to think about an economy with diverse beliefs, why such an economy presents a complex challenge to central bank policy and why such complexity is absent from a model with a homogenous belief. The starting point for such intuition is the recognition that diverse expectations introduce into the market complex interactions. Expectations alter the motivation of agents to consume, work, produce, borrow and invest in assets in order to act upon their beliefs. While a technology shock increases present and future output, change in expectations can also change demand and output today. However, changed expectations entails a cascade of other effects such as expected higher wage rate in the future which can reduce the supply of labor today, raising wage rate and marginal cost today and these might lower output today but increase it in the future. Some effects of expectations are realized through borrowing by agents with diverse beliefs. In fact I note that although central bank policy acts on financial assets and borrowing, agents' borrowing plays no role in the dynamics of a representative household model with RE since such an agent does not borrow. In (31a)-(31d) market belief is key to the expectational complexity of the micro equilibrium. How does it work?

Changes in state of belief change individual motives for allocating consumption and work between today and the future, altering the supply of labor, consumption demand and borrowing. This indicates that change in expectations impacts equilibrium wage rate, employment and output. But policy parameters also aim to change the motive for intertemporal allocation of consumption and labor! Hence, expectations may amplify, negate or distort the effect of policy. In addition, changed expectations interact with other shocks resulting in greater complexity of causes for changes in output; increased interest rate in response to increased

Policy Frontier of Output and Inflation for (u,v) model under Rational Expectations

Figure 1.2 Under Taylor Rule and (v, u) Shocks with σ_u=0.018

$1.1 \leq \xi_\pi \leq 10$ and $0 \leq \xi_y \leq 5$

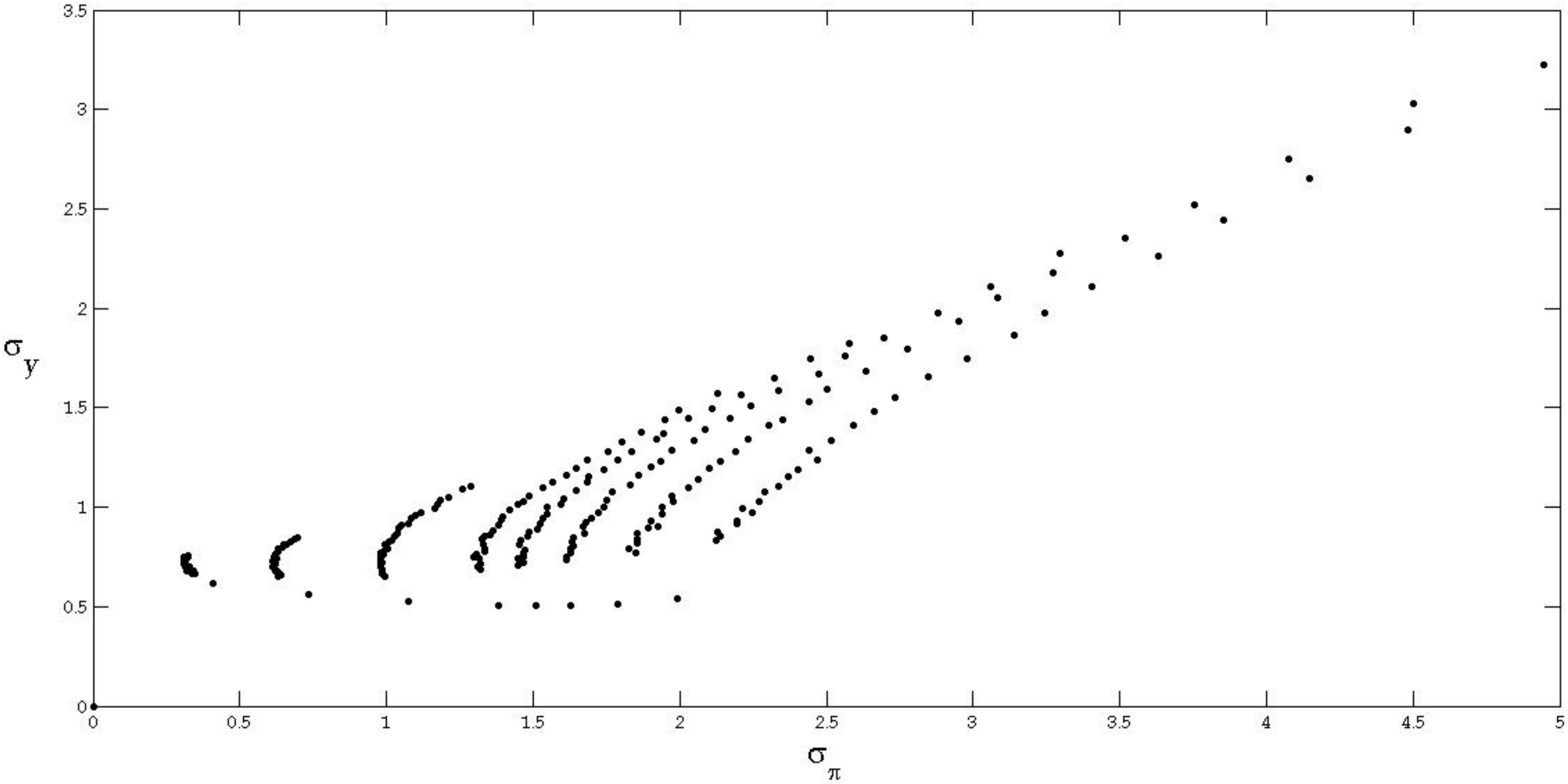
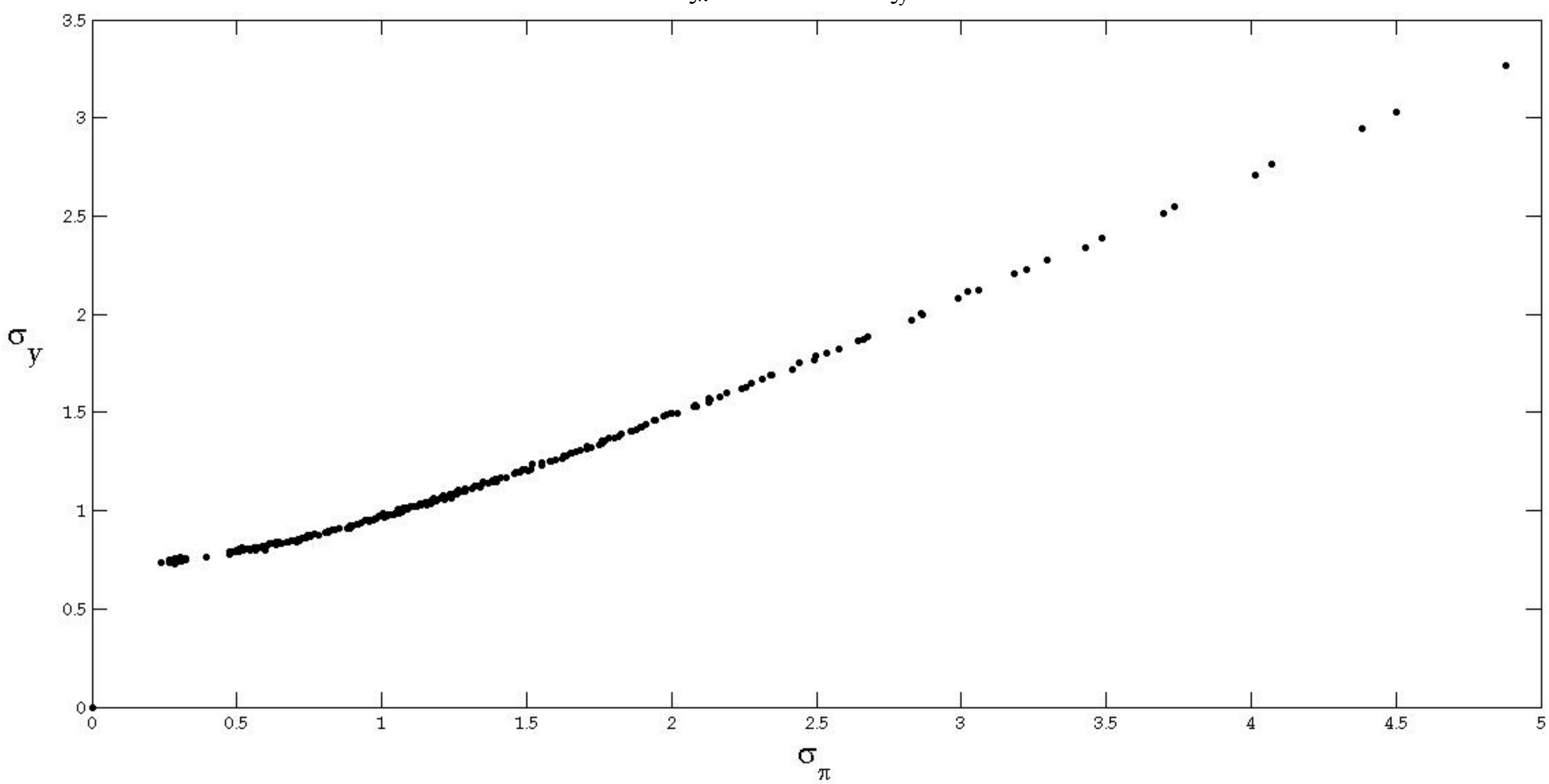


Figure 1.3 Under Output Gap Rule and (v, u) Shocks with σ_u=0.018

$1.1 \leq \xi_\pi \leq 10$ and $0 \leq \xi_y \leq 5$



income might be self defeating for policy stabilization. These are highly non-linear interaction effects which change the simple picture outlined in Table 1.1 through their impact on the shape of the Equilibrium Manifold. As policy parameters change, the equilibrium map changes, altering the parameter values of (B_y, B_π) in (31a)-(31d) hence the values of (σ_y, σ_π). Based on results we have seen up to now even with models under RE, the presence of diverse beliefs about state variables impacts the economy in a manner which is analogous to economies with multiple shocks but with a particular structure and particular response to changes in policy. Hence, it may be useful to keep in mind the following observations:

(i) The effect of policy may not be monotonic in policy parameters: a policy parameter may have a threshold at which its effect is reversed hence increased ξ_y or ξ_π may increase or decrease σ_y or σ_π.

(ii) Impulse response of key variables to shocks may change direction at different policy parameters and exhibit an increase or decrease due to expectational effects.

The simulations in the next section show that all of the above actually takes place normally.

Table 1.3: RBC Volatility and Monotonicity under Rational Expectations

Standard RBC with $\sigma_\zeta = 0.003$

σ_y		ξ_y 0.0	0.6	1.2	1.5	30	σ_π		ξ_y 0.0	0.6	1.2	1.5	30
ξ_π	1.0	0.55	0.21	0.13	0.11	0.01	ξ_π	1.0	0.50	1.44	1.66	1.71	2.02
	1.6	0.69	0.53	0.44	0.39	0.04		1.6	0.09	0.52	0.81	0.89	1.91
	2.2	0.71	0.62	0.54	0.51	0.08		2.2	0.05	0.33	0.53	0.63	1.81
	2.8	0.72	0.63	0.58	0.56	0.11		2.8	0.03	0.23	0.39	0.47	1.72
	30	0.73	0.72	0.72	0.72	0.54		30	0.01	0.02	0.03	0.04	0.56

Finally, recall the RBC approach assumed technology shocks with $\sigma_\zeta = 0.0072$. Strong objections were raised against this measure and persuasive case was made supporting the view that much of this residual is not technology. It is suggested it is at most $\sigma_\zeta = 0.003$ hence I set the value of $\sigma_\zeta = 0.003$. Table 1.3 shows the well known fact that under RE with a single technology shock and $\sigma_\zeta = 0.003$ the models's business cycles volatility virtually disappears. I show later that when diverse beliefs are present, volatility is amplified and fluctuations contain a major component of endogenous uncertainty due to market belief. Under such conditions the model exhibits realistic volatility and public stabilization policy of output and inflation becomes relevant. But then, any central bank policy has an important objective of stabilizing the volatility amplification effect of market expectations, in addition to the effects of exogenous shocks or sticky prices.

Turning now to economies with diverse beliefs, I allow $\xi_y < 0$ as long as the determinacy conditions hold. Tables 2.1 - 2.3 report results of simulating the (v, u) diverse belief model (30a)-(30c) with standard realistic belief parameter values, most of which were motivated earlier:

$\lambda_Z = 0.6$, $\lambda_u = 0.8$, $\lambda_Z^v = 0.15$, $\lambda_Z^u = 0.1$, $\lambda_Z^g = 0.05$, $\lambda_v^g = 1$, $\lambda_u^g = 1$, $\sigma_\zeta = 0.003$, $\sigma_g = 0.003$, $\sigma_u = 0.006$.

Estimation of the policy rule for the US by H. Wu lead to parameter values of $\lambda_u = 0.8$, $\sigma_u = 0.006$.

TABLES 2.1-2.3 PLACE HERE

(see next page)

Tables 2.1-2.3 show a more complex pattern of non-monotonicity than in Table 1.2 with several reversals in the effect of policy. Starting with σ_y in Table 2.1 and confining discussion to determinacy region, note first the local minimum over ξ_y around $-0.4 \le \xi_y \le -0.2$, the maximal ridge over ξ_y and over ξ_π (painted in green) starting from $\xi_y = -0.2$ and stretching along a semi-diagonal of rising ξ_y and ξ_π. One can then identify three σ_y response regions as follows:

Region 1- $\xi_y < -0.4$: σ_y falls as ξ_y rises and falls as ξ_π rises;

Region 2- $\xi_y \ge -0.4$ and below the maximum with respect to ξ_π: σ_y increases as ξ_y rises and falls as ξ_π rises;

Region 3- $\xi_y \ge -0.4$ and above the maximum with respect to ξ_π: σ_y decreases as ξ_y rises and rises as ξ_π rises.

Observe first that any (ξ_y, ξ_π) in region 1 is *inefficient* since both (σ_y, σ_π) in that region can be reduced by raising both (ξ_y, ξ_π). Also, the combination of Regions 1 and 3 are analogous to the two regions in Table 1.2 hence the impact of diverse beliefs is the new Region 2. In this region an aggressive output stabilization policy using larger values of ξ_y is *self defeating* since it increases the volatility of output rather than decrease it.

Table 2.2 reports the response of σ_π. It is clear increased ξ_π lowers σ_π *in all cases*. As we vary ξ_y, σ_π attains a minimum around $-0.2 \le \xi_y \le 0.2$ (minimal σ_π highlighted in yellow): for larger values of ξ_y, σ_π rises with ξ_y while for smaller values σ_π falls with ξ_y. The minimal values of σ_y and σ_π in Tables 2.1 and 2.2 occur at different values of (ξ_y, ξ_π) hence these minimal values cannot be attained *simultaneously*. However, considering (σ_y , σ_π) one notes that any policy choice in the combined Regions 1 and 2 is dominated by the following simple policy: select ξ_y so as to minimize σ_y and select ξ_π as aggressive as politically feasible. This simple anti-inflation policy, using an aggressive single instrument ξ_π in Regions 2, is a powerful tool to reduce the volatility of both (σ_y , σ_π)! Since this policy can be implemented by selecting $\xi_y = 0$, it actually represents the policy of *a central bank with only a single mandate to control inflation*. But recall that Region 2 is the key addition caused by diverse beliefs. Hence, this aggressive anti-inflationary policy in fact counters the volatility caused primarily by market belief. But then what are the limitations of such a policy?

The policy above has two key limitations. First, it has a bounded effect on lowering the volatility of output, as seen at the bottom of Table 2.1. As $\xi_\pi \Rightarrow \infty$, the value of σ_y is bounded away from zero. To reduce output volatility lower than this bound, a policy maker must move to Region 3. In this region aggressive

Table 2.1 Output Volatility In The Two Shocks (u,v) Model Under Diverse Beliefs

σ_y		ξ_y												
		-0.6	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**2**	**3**	**4**	**30**
	1.1	n.a.	n.a.	2.69	2.20	1.86	1.62	1.42	1.26	1.14	0.76	0.56	0.45	0.07
	1.3	n.a.	2.07	1.94	2.04	2.08	2.03	1.98	1.91	1.83	1.45	1.20	1.01	0.19
	1.5	n.a.	1.49	1.56	1.76	1.90	1.98	2.02	2.02	2.01	1.80	1.58	1.39	0.31
ξ_π	**1.7**	n.a.	1.23	1.35	1.54	1.71	1.84	1.94	1.98	1.99	1.95	1.80	1.63	0.42
	1.9	1.26	1.09	1.21	1.37	1.55	1.69	1.81	1.88	1.93	2.01	1.91	1.80	0.53
	2.1	1.11	0.99	1.09	1.26	1.44	1.56	1.69	1.77	1.85	1.99	1.96	1.90	0.62
	2.5	0.98	0.90	0.97	1.11	1.24	1.38	1.48	1.58	1.67	1.94	2.00	1.99	0.80
	30	0.72	0.74	0.73	0.74	0.73	0.72	0.73	0.73	0.73	0.77	0.79	0.81	1.69

Table 2.2 Inflation Volatility In The Two Shocks (u,v) Model Under Diverse Beliefs

σ_π		ξ_y												
		-0.6	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**2**	**3**	**4**	**30**
	1.1	n.a.	n.a.	3.47	5.40	8.10	10.12	11.56	12.70	13.59	16.10	17.30	17.92	20.32
	1.3	n.a.	3.28	2.08	2.03	3.23	4.56	5.80	6.95	7.92	11.22	13.26	14.53	19.67
	1.5	n.a.	2.11	1.49	1.28	1.77	2.59	3.48	4.32	5.12	8.30	10.50	12.05	18.85
ξ_π	**1.7**	n.a.	1.56	1.20	0.97	1.17	1.68	2.32	2.95	3.58	6.39	8.51	10.07	18.44
	1.9	1.41	1.23	1.00	0.81	0.87	1.19	1.65	2.15	2.66	5.08	7.03	8.66	17.76
	2.1	1.15	1.02	0.84	0.70	0.70	0.90	1.24	1.62	2.05	4.10	5.88	7.46	17.18
	2.5	0.84	0.76	0.66	0.56	0.53	0.60	0.79	1.03	1.32	2.88	4.36	5.69	16.23
	30	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.06	0.08	0.11	1.76

Table 2.3 Individual Consumption Volatility In The Two Shocks (u,v) Model Under Diverse Beliefs

σ_c		ξ_y												
		-0.6	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**2**	**3**	**4**	**30**
	1.1	n.a.	n.a.	4.96	22.97	25.34	136.42	56.32	58.51	55.39	141.50	94.71	152.18	107.70
	1.3	n.a.	10.70	21.81	2.23	14.83	13.71	42.35	45.62	30.39	54.75	59.70	92.42	191.10
	1.5	n.a.	32.61	9.60	4.46	3.49	14.65	16.85	15.72	26.94	36.85	58.87	46.81	195.09
ξ_π	**1.7**	n.a.	16.37	15.92	6.71	3.54	2.27	6.25	9.76	26.24	38.18	47.93	121.68	66.91
	1.9	10.88	23.81	10.70	15.67	4.65	2.25	2.81	6.42	6.30	23.51	34.61	76.76	148.68
	2.1	11.61	9.49	9.82	16.77	7.74	4.81	1.98	2.31	7.56	20.43	35.26	35.53	61.10
	2.5	26.32	16.99	23.45	12.45	7.25	4.88	3.35	5.24	1.67	10.97	21.25	21.81	101.71
	30	7.52	25.92	11.65	8.03	8.57	14.80	21.94	14.57	19.42	6.30	9.39	16.80	4.93

Note: In the region below the bold lines the parameters satisfy Blanchard-Kahn conditions.

output stabilization policy is effective and large values of ξ_y do suppress output volatility but at the cost of higher inflation volatility. It is interesting that the pattern of (σ_y, σ_π) trade-off in region 3 is similar to the pattern under RE in Table 1.1. What is new here is the fact that there is a policy choice between Region 2 and Region 3. It is a trade-off between aggressive inflation stabilization by a single mandate central bank in Region 2 and dual stabilization policies in Region 3. Hence it represent two different but efficient visions of central bank policy. I examine this (σ_y, σ_π) trade-off between Regions 2 and 3 in the next two Sections.

The second limitation of an anti-inflationary policy in Region 2 is unique to heterogenous economies and does not exist in a single agent economy: the volatility of financial markets and *individual* consumption. Table 2.3 reports the effect of policy on volatility of individual consumption σ_c. This volatility cannot be computed from the macro model; it must be computed from the micro-economic equilibrium in which individual agents are symmetric. These agents hold diverse beliefs and borrow or lend to act upon these beliefs. Monetary policy has an impact on their choices and variable interest rates interact with private expectations to create a difference between volatility of *individual* and *aggregate* consumption[9]. Note the semi-diagonal configurations where σ_c is minimized. One can show that such a minimum occurs when volatility of the real rate is minimized. But an aggressive monetary policy to stabilize (σ_y, σ_π) does not aim to reduce the volatility of the real rate. Indeed, volatility of the real rate is a key tool of inflation stabilization when utilizing $\xi_\pi > 1$ policy. As can be seen from Table 2.3 a policy to stabilize the real rate requires a delicate balance between large values of ξ_π and large values of ξ_y. Consequently, one can see that a single mandate central bank that selects $\xi_y = 0$ and stabilizes inflation with a large value of ξ_π will destabilize individual consumption. Welfare considerations thus imply that monetary policy must face the complex trade-off among $(\sigma_y, \sigma_\pi, \sigma_c)$: volatility of aggregate output, inflation and individual consumption! Since σ_c results from high volatility in the bond market, one must consider σ_c not only as volatility of individual consumption but also as volatility of financial markets in general. It is an undisputed fact that all central banks are concerned with volatility of financial markets! This discussion also questions the common view that volatility of aggregate consumption and output should be the only objective of policy, as demonstrated by the pointless discussion that followed Lucas (1987),(2003).

[9] The reader may find elsewhere (see Kurz (2010) and Kurz and Motolese (2011)) detailed explanation of why it is not incentive compatible to have markets for claims which are contingent on future market belief. Here I note briefly that market belief must be computed using data on surveys of individual forecasts and the existence of such markets will create a public motive to distort the reported forecasts. The portion of the population which is short will have an incentive to report so as to lead to computed low level of market belief and the portion which is long will have the incentive to report the opposite. No court can rely upon such information to resolve legal disputes about financial obligations. Due to market incompleteness agents cannot reduce consumption volatility by trading in markets for contingent claims.

Without exhibiting an additional table one can see that the volatility of individual consumption is associated with the volatility of borrowing and bond holdings. Hence, consumption volatility which results from volatility of the real rate is a proxy for financial markets volatility. It explains that political resistance to aggressive use of anti-inflation instrument ξ_π is rooted in the fact that aggressive use of ξ_π entails volatile financial markets and high volatility of interest rates and individual consumption. The next two Sections will attempt to clarify this complex tradeoff.

6.4 *Explaining Tables 2.1-2.3:Interaction of Policy with Market Belief and the Composition Effect of Expectations*

Conclusions drawn from Tables 2.1-2.3 are central and I now explain that they result from interaction between policy and market beliefs. Tables 2.1-2.2 show that under the impact of diverse beliefs a central bank has a choice of acting as a single mandate bank in Region 2 or as a dual mandate central bank in Region 3. On the boundary between these two regions output volatility attains maximal levels identified in Table 2.1 by the semi-diagonal ridge marked in the Table with green. I claim that in Region 2 the aggressive (select ξ_π as large as politically feasible!) anti-inflation effort of the central bank is directed squarely against the amplification effect of market belief on the volatility of output and inflation. The single mandate central bank can "crush" the effects of market expectations by aggressive ξ_π policy and by setting $\xi_y = 0$.

To prove these statements one must examine how policy alters the equilibrium map since this will reveal how policy alters the impact of expectations on each endogenous variable. To that end note that in the linearized economy equilibrium variables are linear functions of state variables. By (29a)-(29f) the map is

(29a) $$\hat{c}_t^j = A_y^Z Z_t + A_y^v v_t + A_y^u u_t + A_y^b \hat{b}_{t-1}^j + A_y^g g_t^j$$

(29d) $$\hat{y}_t = (A_y^Z + A_y^g) Z_t + A_y^v v_t + A_y^u u_t$$

(29e) $$\hat{\pi}_t = (A_\pi^Z + A_\pi^g) Z_t + A_\pi^v v_t + A_\pi^u u_t.$$

Hence, fluctuations of the aggregates $(\hat{y}_t, \hat{\pi}_t)$ are determined by fluctuations of Z, v and u. The effect of market belief Z on σ_y is measured by $|A_y^Z + A_y^g|$ and its effect on σ_π is measured by $|A_\pi^Z + A_\pi^g|$. Table 2.4 reports how policy alters the term $(A_y^Z + A_y^g)$ and Table 2.5 reports how policy changes $(A_\pi^Z + A_\pi^g)$.

A comparison of Table 2.1 with Table 2.4 shows that σ_y is maximized exactly by the same policy parameters which maximize $|A_y^Z + A_y^g|$, along the semi-diagonal ridge in Table 2.1. Also, a comparison of Table 2.2 with Table 2.5 shows that inflation volatility σ_π is minimized exactly where $(A_\pi^Z + A_\pi^g)$ takes the value 0 and changes sign from positive to negative. Finally, regardless of the value of ξ_y chosen by the central bank, an aggressive anti-inflation policy with $\xi_\pi = 30$ eliminates the effect of market belief on (σ_π, σ_y) by "crushing" $|A_y^Z + A_y^g|$ and $|A_\pi^Z + A_\pi^g|$ towards 0. Hence, a central bank that is concerned only with a stabilization of the aggregates (σ_π, σ_y) and who chooses policy in Region 2 may as well select $\xi_y = 0$ and use the one instrument ξ_π as aggressively as politically feasible. This demonstrates that in Region 2 stabilization of the aggregates (σ_π, σ_y) is an effort by the central bank to counter the impact of market expectations.

The direct effect of market belief Z on output and inflation is noteworthy. *Under all efficient policy parameters, a market belief in a better future economic conditions increases today's output since it results in a higher level of employment.* However, the effect on prices and inflation is more complex. Table 2.5 shows there are policy parameters that lead to lower nominal wage and lower inflation and others that lead to higher nominal wage and higher inflation.

TABLES 2.4-2.7 PLACE HERE

(see next page)

Table 2.3 clarifies that the social cost of aggressive anti-inflation policy is high volatility of individual consumption and financial markets associated with volatile interest rates. Individual consumption is more volatile than mean consumption (= aggregate output) and welfare considerations suggest that a central bank should not ignore the effect of policy on individual consumption, regardless of how important the aggregates are. The effect of belief on individual consumption is complex: A_y^g measures the effect of g_t^i on agent i's consumption at date t and A_y^Z measures the effect of market belief Z_t on agent i's consumption at date t, and these two measure different quantities. Although both g_t^i and Z_t measure beliefs at date t about a better future conditions of the economy at t+1 (g_t^i is of measure of agent i and Z_t is mean of the market) there is a crucial difference between them with regard to the way they impact individual consumption:

- g_t^i impacts date t agent i's consumption (i.e. A_y^g) via its effect on the agent's expectation of *date t+1 state variables* that define t+1 wage rate, income, inflation, interest rate and his own consumption. For example, in (29a)-(29e) i's forecasts of $(Z_{t+1}^i, v_{t+1}^i, u_{t+1}^i)$ depend upon g_t^i in accord with his perception.
- Z_t impacts *date t* individual consumption (i.e. A_y^Z) via its effect on date t endogenous market variables which impact the agent's budget constraint at date t. One should think of the effect of Z_t on individual consumption in the same way market prices impact consumption demand.

A comparison of Table 2.3 with Tables 2.6 and 2.7 shows the interaction of belief with policy is complex. Generally speaking the two equilibrium parameters A_y^Z and A_y^g *are minimized at approximately the same policy parameter configurations* when each of them is close to 0 on its own. Hence, the lowest volatility of individual consumption occurs at the policy parameters which minimize these two equilibrium parameters.

Table 2.4 $(A_y^z + A_y^g)$ in the Two Shocks (u,v) Model Under Diverse Beliefs

								ξ_y						
		-0.6	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**2**	**3**	**4**	**30**
	1.1	n.a.	n.a.	4.52	4.96	4.57	4.10	3.67	3.31	3.01	2.04	1.54	1.23	0.20
	1.3	n.a.	1.20	4.07	5.21	5.61	5.67	5.58	5.41	5.21	4.21	3.46	2.92	0.56
	1.5	n.a.	1.25	3.29	4.46	5.12	5.48	5.66	5.72	5.70	5.18	4.55	4.01	0.90
ξ_π	**1.7**	n.a.	1.08	2.71	3.80	4.53	5.02	5.34	5.54	5.65	5.58	5.16	4.70	1.22
	1.9	-0.95	0.93	2.29	3.29	4.02	4.55	4.94	5.22	5.42	5.70	5.49	5.15	1.52
	2.1	-0.72	0.81	1.98	2.89	3.59	4.14	4.56	4.88	5.13	5.66	5.64	5.42	1.79
	2.5	-0.48	0.64	1.56	2.32	2.95	3.47	3.90	4.26	4.55	5.40	5.66	5.65	2.30
	30	-0.02	0.04	0.10	0.16	0.21	0.27	0.33	0.38	0.44	0.70	0.95	1.19	4.60

Table 2.5 $(A_\pi^z + A_\pi^g)$ in the Two Shocks (u,v) Model Under Diverse Beliefs

								ξ_y						
		-0.6	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**2**	**3**	**4**	**30**
	1.1	n.a.	n.a.	1.35	-12.67	-21.68	-27.84	-32.28	-35.64	-38.26	-45.76	-49.32	-51.39	-58.04
	1.3	n.a.	6.10	2.20	-2.83	-7.66	-11.97	-15.74	-19.02	-21.87	-31.83	-37.66	-41.47	-56.10
	1.5	n.a.	3.98	2.13	-0.53	-3.43	-6.31	-9.05	-11.62	-14.00	-23.36	-29.68	-34.16	-54.25
ξ_π	**1.7**	n.a.	2.98	1.90	0.25	-1.67	-3.70	-5.73	-7.72	-9.63	-17.83	-23.97	-28.61	-52.49
	1.9	2.43	2.39	1.68	0.55	-0.81	-2.30	-3.85	-5.42	-6.97	-14.03	-19.75	-24.30	-50.82
	2.1	2.04	2.00	1.49	0.68	-0.34	-1.48	-2.70	-3.96	-5.23	-11.31	-16.54	-20.89	-49.22
	2.5	1.53	1.50	1.21	0.73	0.10	-0.63	-1.44	-2.29	-3.18	-7.75	-12.06	-15.90	-46.25
	30	0.08	0.08	0.08	0.08	0.08	0.08	0.07	0.07	0.06	0.02	-0.03	-0.10	-4.77

Table 2.6 A_y^z in the Two Shocks (u,v) Model Under Diverse Beliefs

								ξ_y						
		-0.6	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**2**	**3**	**4**	**30**
	1.1	n.a.	n.a.	0.82	16.98	26.44	32.64	37.00	40.24	42.73	49.76	53.03	54.92	60.90
	1.3	n.a.	-9.96	-1.71	5.49	11.48	16.42	20.54	24.00	26.95	36.81	42.37	45.93	59.19
	1.5	n.a.	-8.07	-3.16	1.48	5.69	9.47	12.83	15.83	18.51	28.43	34.73	39.06	57.55
ξ_π	**1.7**	n.a.	-7.47	-3.95	-0.54	2.68	5.68	8.45	11.00	13.34	22.61	29.02	33.67	55.99
	1.9	-9.88	-7.19	-4.44	-1.74	0.85	3.32	5.64	7.83	9.88	18.37	24.61	29.33	54.49
	2.1	-9.24	-7.02	-4.77	-2.54	-0.38	1.71	3.71	5.61	7.42	15.15	21.11	25.79	53.05
	2.5	-8.49	-6.85	-5.19	-3.54	-1.91	-0.33	1.22	2.71	4.16	10.63	15.96	20.36	50.36
	30	-6.56	-6.47	-6.38	-6.30	-6.21	-6.12	-6.04	-5.95	-5.86	-5.43	-4.99	-4.56	5.82

Table 2.7 A_y^g in the Two Shocks (u,v) Model Under Diverse Beliefs

								ξ_y						
		-0.6	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**2**	**3**	**4**	**30**
	1.1	n.a.	n.a.	3.70	-12.01	-21.87	-28.54	-33.33	-36.93	-39.72	-47.71	-51.49	-53.68	-60.70
	1.3	n.a.	11.16	5.79	-0.28	-5.86	-10.75	-14.96	-18.59	-21.74	-32.60	-38.91	-43.01	-58.62
	1.5	n.a.	9.32	6.45	2.98	-0.57	-3.98	-7.17	-10.12	-12.81	-23.25	-30.18	-35.05	-56.65
ξ_π	**1.7**	n.a.	8.55	6.66	4.34	1.85	-0.66	-3.11	-5.46	-7.69	-17.03	-23.86	-28.96	-54.77
	1.9	8.93	8.12	6.74	5.03	3.17	1.24	-0.70	-2.61	-4.46	-12.67	-19.12	-24.19	-52.97
	2.1	8.52	7.83	6.76	5.44	3.97	2.43	0.85	-0.73	-2.29	-9.49	-15.47	-20.37	-51.26
	2.5	8.01	7.49	6.75	5.86	4.86	3.80	2.68	1.54	0.39	-5.23	-10.30	-14.70	-48.06
	30	6.54	6.51	6.48	6.45	6.42	6.39	6.36	6.33	6.30	6.13	5.95	5.75	-1.22

Note: In the region below the bold lines the parameters satisfy Blanchard-Kahn conditions.

This minimum of A_y^Z and A_y^g is very different from the policy configurations which minimize either mean consumption (= output) volatility or inflation volatility. Hence, an important policy trade-off is implied!

The Composition Effect of Expectations

The two distinct effects of Z and g on individual consumption are interesting and go to the heart of a diverse beliefs theory. Table 2.7 shows policy alters the effect of g on individual consumption in a dramatic manner: a policy with small ξ_y and large ξ_π (left low corner of Table 2.7) increases individual consumption if g > 0. On the other hand, a policy with large ξ_y and small ξ_π (right up corner of Table 2.7) changes the causation and decreases consumption if g > 0. To explain it recall my earlier argument that changed individual's belief has multiple effects. For example, $g_t^i > 0$ increases expected date t+1 wage by agent i and such a change can lead to two possible outcomes. According to one, at date t the agent increases borrowing and work effort in order to increase date t consumption due to the t+1 income effect involved. Alternatively, due to intertemporal substitution the agent may increase labor supply at date t+1 and take more leisure at date t thus lowering his income and consumption at date t. The net effect depends upon the interest rate and *here is where policy interacts with expectations*. A large ξ_y means that increased work effort increases output and this leads to higher interest rate. The higher the central bank sets ξ_y the higher is the interest rate resulting from higher output hence the larger is the incentive to save at t and increase consumption at t+1. This explains why values are *negative* on the upper right side of Table 2.7 and *positive* in the opposite side of the table.

It may appear surprising that in Table 2.6 the effect of market belief Z on individual consumption *is exactly the opposite of the effect of g*. But this is the composition effect of expectations! To understand it return to the previous paragraph and suppose a (small ξ_y, large ξ_π) policy on the left low corner of Table 2.7 causes agent i to respond to $g_t^i > 0$ by increased desired borrowing and work effort, thus increasing date t consumption. Such a decision is made given all *endogenous variables being equal*. But now if a majority of j agents hold similar beliefs with $g_t^j > 0$ then $Z_t > 0$. Table 2.6 shows that such increased date t aggregate desire to borrow, desired work effort and demand for consumption change *date t* market variables (wage rate, interest rate, inflation rate etc.) in a manner that works to frustrate the desires of agent i. In other words, if $g_t^i > 0$ it is easier for agent i to act upon his expectations if at the same time $Z_t \le 0$ since under $Z_t \le 0$ the market conditions are more favorable to the belief of agent i. *Simply put, if a majority of agents hold similar beliefs as you (e.g.* $Z_t > 0$ *and you are i with* $g_t^i > 0$*), they frustrate your efforts to act upon your belief by turning the market against you because in that case too many people make similar forecasts since (*$g_t^j > 0$ *for* $j \neq i$*) and, acting on their belief, they desire at t the same thing as you do!* Parameter A_y^Z measures the net effect the market belief (i.e. Z_t) on the consumption of agent i. This effect frustrates the effort of the individual agent when $g_t^i > 0$ *but bolsters his effort when* $g_t^i < 0$. The direction of the effort depends upon policy.

6.5 *Policy Tradeoff Under Diverse Beliefs: Should Central Banks Have A Dual Or Single Mandate?*

I now turn to a study of the policy trade-offs implied by Tables 2.1-2.3 under a Taylor rule with a shock u in the IS curve and Figures 2.1-2.3 present frontiers generated by the two shock model (u ,v) under this rule. Aggregate volatilities (σ_π, σ_y) in Figure 2.1 are generated by policies in all three Regions of Table 2.1 discussed below. The scatter of points in the upper left, where (σ_π, σ_y) decline together towards the left reflect inefficient policies when ξ_y are lower than values that minimize output or inflation volatilities in region 1 of Table 2.1. One may ignore these policies. The solid mass of points on the left results from aggressive anti-inflation policy in Region 2 of Table 2.1. As explained earlier, this segment reflects the actions of a central bank with a single mandate to fight inflation aggressively. By selecting $\xi_y = 0$ and as large value of ξ_π as politically feasible the policy, in fact, moves the economy down the left segment along which both (σ_π, σ_y) fall and the policy stabilizes both inflation and aggregate output. With this policy, output volatility has a positive lower bound which can be large and aggressive anti inflation policy has crucial cost I shortly explore.

The Frontier's negatively sloped upper segment in Figure 2.1, along which a trade-off between σ_y and σ_π is exhibited, is generated by policies in Region 3 and reflect options available to a dual mandate central bank. This downward sloping frontier is analogous to the frontier implied by Table 1.1 and is compatible with the results of Taylor (1979), Fuhrer (1994), Ball (1999), Rudebusch and Svensson (1999) and others. On this frontier one can show $\sigma_y \Rightarrow 0$ as $\xi_y \Rightarrow \infty$ but *inflation volatility rises*. But Figure 2.1 demonstrates a deeper (σ_π, σ_y) policy choice between the negatively sloped part of the frontier, employed by a dual mandate bank that selects (σ_π, σ_y) by fine tuning (ξ_π, ξ_y), and the positively sloped left part that results from a choice of a single mandate central bank who follows an aggressive anti-inflation policy. For different shocks and different parameter values the lower bound of σ_y under such a policy could vary a great deal.

To help the reader distinguish between the three Regions discussed, Figure 2.2 records the policy outcomes of one row of Tables 2.1-2.2 defined for $\xi_\pi = 1.9$ and $-0.9 \le \xi_y \le 5$. One reads Figure 2.2 by noting that the points are ordered as one reads Table 2.1-2.1 for $\xi_\pi = 1.9$ and increasing values of ξ_y, starting at -0.9 (the highest point in Figure 2.2). For increasing values of ξ_y the points traced in Figure 2.2 decline from the very top to the bottom. These are all associated with inefficient policies resulting from $-0.9 \le \xi_y \le -0.4$ at the end of which the minimal value of σ_y is reached. Then, for $-0.4 \le \xi_y \le 0.0$ there is a real (σ_π, σ_y) trade-off: σ_y is rising and σ_π falling. This tradeoff is seen in the short negatively sloped section of the plotted curve on the left side. For $\xi_y > 0$ the policy enters Region 2 and as we move to the maximal value of σ_y we reach the green labeled ridge in Table 2.1 where σ_y attains a maximum at approximately $\xi_y = 2$. For all values $\xi_y > 2$ the policy enters Region 3 at which point a (σ_π, σ_y) trade-off exists as in Table 1.1 under RE. The collection of points in Figure 2.1 is, in fact, the union of all points like those in Figure 2.2, for $1.1 \le \xi_y \le 5$.

Figure 2.1-2.3 PLACE HERE

(See next page)

I now turn to the more complex choice faced by a central bank, which is the volatility of individual consumption. Figure 2.3 records the collection of feasible inflation and individual consumption volatilities which are implied by policy choices (ξ_y, ξ_π) in the specified ranges, under a Taylor rule with a shock u. It is rather surprising that the efficient frontier of (σ_π, σ_c) exhibits smooth concave policy trade-off implicit in Tables 2.2-2.3. I have noted that welfare considerations suggest the policy trade-off between σ_π and σ_c should be of interest. I have also pointed out that Table 2.3 shows that the policy $\xi_y = 0$ and large ξ_π will destabilize σ_c and I add the claim that *efficient choices of* (σ_π, σ_c) *are typically attained with moderate values of the* (ξ_y, ξ_π) *instruments* while aggressive ξ_π do not contribute to the frontier. This last fact cannot be seen in Figure 2.3 since it does not identify parameters that induce the outcomes. I will return to this question in Figures 3 and 4 to demonstrate the proposition asserted.

Up to now I have discussed policy rule (30c) with weights on output and inflation only. One of my conclusions is that in an economy with diverse beliefs the effect of such instruments is complex, non-monotonic and with thresholds which are difficult for a central bank to assess with precision. The policy implication is that output stabilization is a difficult task which requires a bank to have a precise knowledge of the true response surface. It is thus not surprising that differences exist among central banks with respect to the goal of output stabilization. If, however, a central bank is committed to output stabilization, is there a more efficient way to attain it? I will argue in the next Section that to attain output stabilization the bank needs to have a detailed knowledge of the *causes* for output volatility since with such a knowledge, targeting the causes of output volatility is more efficient than targeting output itself.

6.6 Targeting Market Belief and Other Causes of Volatility Instead of Output

It is useful to start with a more formal clarification of why stabilizing output is complex. To that end recall the earlier remarks about interaction between expectations and policy. I have pointed out in discussing Table 1.3 that with $\sigma_\zeta = 0.003$ technology has a small effect on volatility. It is also clear that policy cannot change private expectations of $(v_{t+1}, u_{t+1}, Z_{t+1})$ but it can change private cost of acting on such belief.

Figure 2.1 Policy Frontier of Output and Inflation (σ_y, σ_π) for (u,v) model: -0.9≤ ξ_y ≤5, 1.1≤ ξ_π ≤5

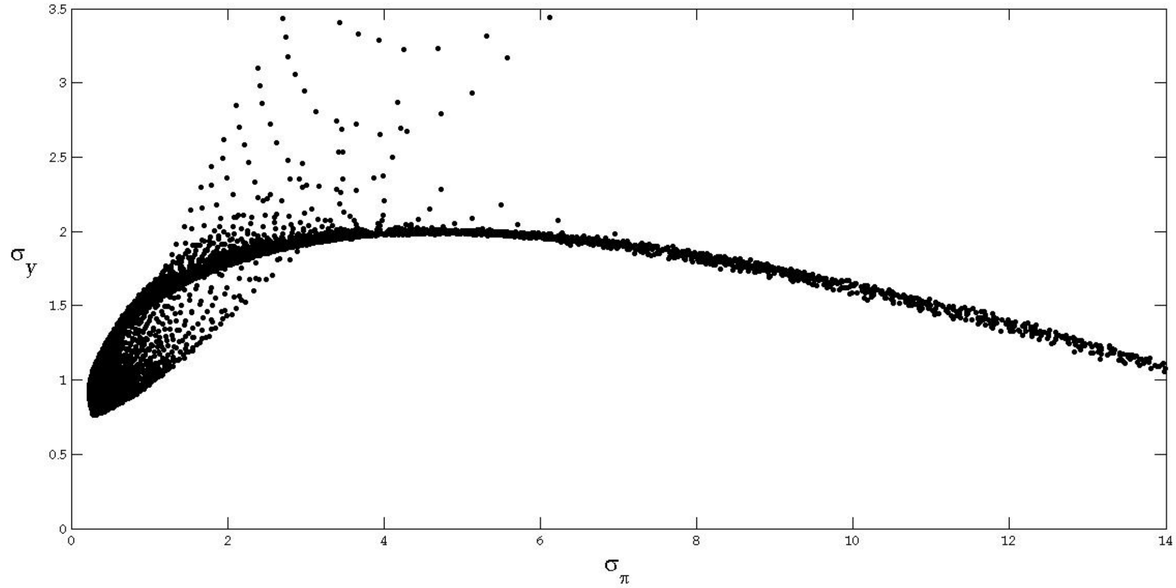


Figure 2.2 Policy Frontier of Output and Inflation (σ_y ,σ_π) for (u,v) model: -0.9≤ ξ_y≤ 5, ξ_π =1.9

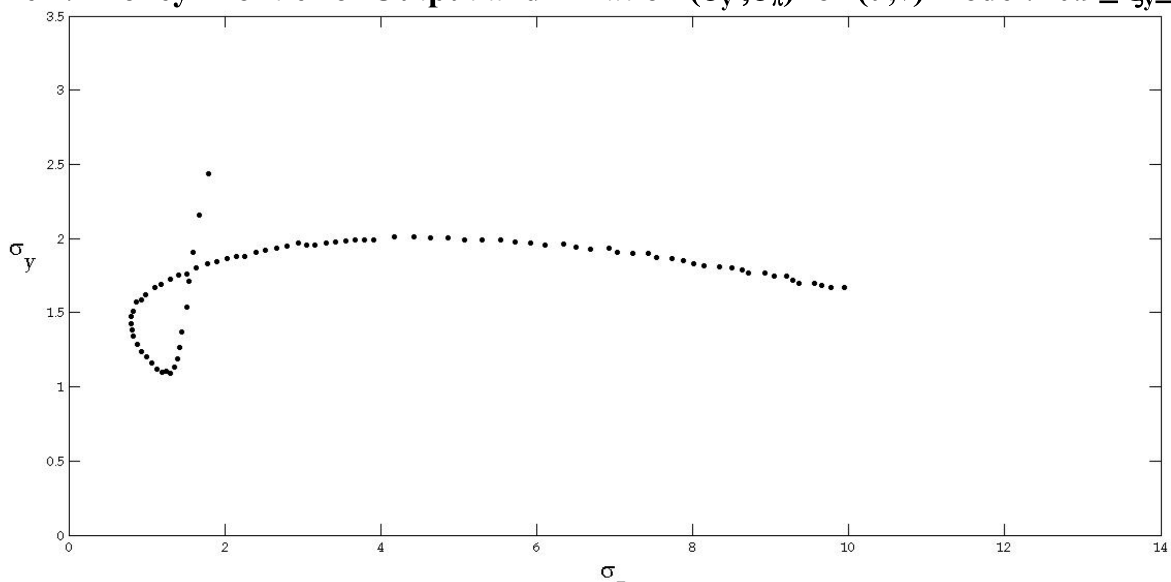
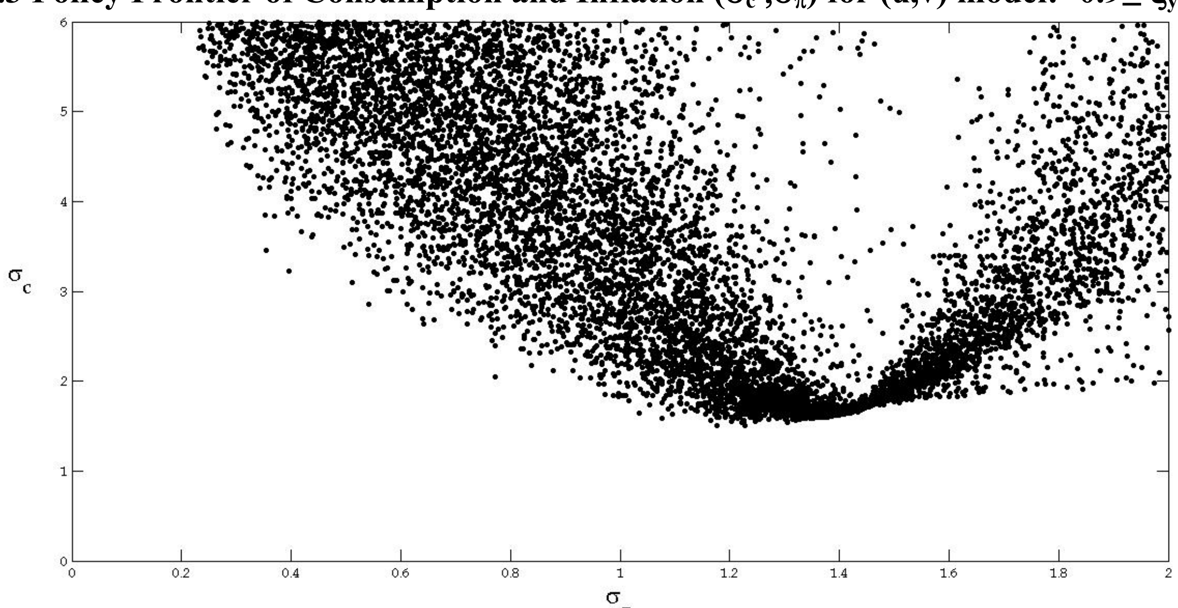


Figure 2.3 Policy Frontier of Consumption and Inflation (σ_c ,σ_π) for (u,v) model: -0.9≤ ξ_y ≤5, 1.1≤ ξ_π ≤5



Equilibrium output is an outcome of several individual motives such as a motive to consume, to supply labor and maximize profits, all of which are influenced by exogenous shocks and expectations. With complex incentives the policy instrument ξ_y acts *jointly* on the multiple effects of shocks and beliefs without a fine distinction between these very different factors. The complexity of these incentives is reflected in the complex curvature of the Equilibrium Manifold as seen in Tables 2.1-2.3. In short, the instrument ξ_y is not fine enough to control separately the diverse effects of exogenous shocks and market belief. Indeed, this "bundling" results in distorting the surface which measures the effect of ξ_y. Instead of targeting output volatility the central bank should, if possible, target the causes of output volatility which are the state variables of the economy since this is a much finer policy tool.

A comment about the shock u_t is in order. I interpret u_t to reflect persistent deviations from the rule, caused by unusual forces beyond the control of the central bank. This may reflect forces such as financial crises, market crashes, political pressure or simple disagreements among members of the open market committee. More general, it reflects economic or security emergencies resulting in deviations with empirical transitions expressing persistence as in (17b). Hence the shock u_t is peculiar in the fact that *any central bank that could target it will simply not allow it to occur.* Hence, for simplicity I set in this Section u = 0.

I suggest that we examine policies that target market belief and all other state variables instead of output. In this model the state variables are (v, Z) and the rule would therefore be

(37) $$\hat{r}_t = \xi_\pi \hat{\pi}_t + \xi_v v_t + \xi_Z Z_t .$$

Since I have already explored the Taylor rule, I will compare (37) with the rule that targets the output gap:

(37a) Output gap and inflation: $$\hat{r}_t = \xi_\pi \hat{\pi}_t + \xi_y [\hat{y}_t - \frac{1}{\kappa(\sigma + \eta)} v_t] \equiv \xi_\pi \hat{\pi}_t + \xi_y [\hat{y}_t - \hat{y}_t^{f}] .$$

Before presenting simulation results for rule (37)-(37a) I have

Proposition 7: Consider an expanded policy rule $\hat{r}_t = \xi_\pi \hat{\pi}_t + \xi_y \hat{y}_t + \xi_Z Z_t + \xi_v v_t$. Then, the Equilibrium Determinant is independent of (ξ_Z , ξ_v) hence changes in these policy parameters do not alter the singularities of the Equilibrium Manifold. They also have no impact on any of the determinacy conditions.

Since in equilibrium output is a function of state variables, for some pairs (ξ_Z, ξ_v), rules (37) and (37a) have the same feasible outcomes. But feasibility is not efficiency. The reason one should expect (37) to perform better than the Taylor rule or (37a) is that under these two the central bank responds only to output or gap which are complex functions of many state variables. In contrast, (37) enables the bank a fine response to each factors that causes output volatility. Observe that, leaving aside the incorrect argument claiming the bank should target $\hat{y}_t^f = [1/((\sigma + \eta)\kappa)]\, v_t$, under (37a) the bank responds- as in (37) - to $\hat{y}_t^f$ and to v_t but with two policy parameters which are in a fixed proportion. In (37) these proportions are not fixed.

Inspecting the rules leads to two questions. (A) what are the monotonicity properties of (ξ_Z, ξ_v) under (37) in comparison with those of ξ_y under (37a), properties which we have already seen in Tables 2.1 - 2.3? (B) what is the difference between the efficient policy frontiers of the two rules? Table 3 provides result which answer question (A) for rule (37) given $\xi_\pi = 1.5$ but the results are the same for other values of ξ_π.

TABLE 3 PLACE HERE

(see table on the next page)

In Table 3 I highlight in yellow the thresholds of the minimal value of each measure of volatility with respect to ξ_Z and the results in the Table show that the response surface with respect to either v or Z has only one threshold. Hence, the response is monotonic on either side of the threshold. In general, thresholds may be points of minimum relative to which the response, away from the threshold, is monotonic increasing but other thresholds may be maximal points. However, for all thresholds in the space of (ξ_Z, ξ_v) the response of is monotonic as one moves away from the threshold and thus it exhibits relatively simple pattern which permits predictability of response to policy. I suggest such monotonicity is a desirable property. This pattern is simpler than the response in Tables 2.1-2.3 to a central bank that, under the Taylor Rule, targets output.

One may inspect Table 3 with the view of deducing an efficient frontier. σ_y is minimal on a steep semi-diagonal line along which both ξ_v and ξ_Z rise. σ_π exhibits the pattern of a minimum on a relatively steep semi-diagonal line along which ξ_v declines and ξ_Z rises. The pattern of σ_c is similar to inflation volatility but along which ξ_v declines only slowly. This argument shows that, to answer question (B) above, it is not easy to deduce the efficient policy frontier from Table 3. I thus study next the efficient policy frontier by drawing it. This shows that the rule (37) dominates (37a) in both the (σ_π, σ_y) and (σ_π, σ_c) spaces. I then show that the efficient policy frontier for (σ_π, σ_c) employs mostly moderate policies.

Simulations reported in Figures 3.1-3.2 use $1.1 \le \xi_\pi \le 2.1$, $-0.9 \le \xi_y \le 3$, $-10 \le \xi_v \le 10$, $-10 \le \xi_Z \le 10$. Black circles are (σ_π, σ_y) outcomes in Figure 3.1 and (σ_π, σ_c) outcomes in Figure 3.2 while red solid lines

Table 3: Targeting the Causes of Output Volatility Rather than Output
(v model with ξ_π=1.5, ξ_y=0)

σ_y		ξ_z														
		-9	**-5**	**-1**	**-0.8**	**-0.6**	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**5**	**9**
	-1	1.6	1.1	1.5	1.5	1.5	1.6	1.6	1.7	1.7	1.8	1.8	1.8	1.9	2.8	3.9
	-0.6	2.2	1.3	0.9	0.9	0.9	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.2	2.0	3.0
	-0.4	2.6	1.6	0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.9	0.8	0.9	0.9	1.6	2.6
	-0.2	3.0	2.0	1.1	1.0	1.0	0.9	0.9	0.9	0.9	0.8	0.8	0.8	0.8	1.2	2.1
ξ_v	**0**	3.4	2.4	1.4	1.3	1.3	1.2	1.2	1.1	1.1	1.0	1.0	1.0	0.9	0.8	1.7
	0.2	3.8	2.8	1.7	1.7	1.6	1.6	1.5	1.5	1.4	1.4	1.3	1.3	1.2	0.6	1.3
	0.4	4.3	3.2	2.1	2.1	2.0	2.0	1.9	1.8	1.8	1.7	1.7	1.6	1.6	0.7	0.9
	0.6	4.7	3.6	2.5	2.5	2.4	2.4	2.3	2.2	2.2	2.1	2.1	2.0	2.0	1.0	0.5
	1	5.6	4.5	3.4	3.3	3.3	3.2	3.2	3.1	3.1	3.0	2.9	2.9	2.8	1.8	0.7

σ_π		ξ_z														
		-9	**-5**	**-1**	**-0.8**	**-0.6**	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**5**	**9**
	-1	7.4	5.9	4.3	4.3	4.2	4.1	4.0	3.9	3.9	3.8	3.8	3.7	3.6	2.1	1.0
	-0.6	6.2	4.6	3.0	2.9	2.9	2.8	2.7	2.6	2.6	2.5	2.4	2.3	2.3	0.8	1.1
	-0.4	5.5	3.9	2.4	2.3	2.2	2.1	2.0	2.0	1.9	1.8	1.7	1.7	1.6	0.3	1.6
	-0.2	4.8	3.3	1.7	1.6	1.5	1.5	1.4	1.3	1.2	1.2	1.1	1.0	0.9	0.7	2.2
ξ_v	**0**	4.2	2.6	1.1	1.0	0.9	0.8	0.7	0.7	0.6	0.5	0.4	0.4	0.3	1.3	2.9
	0.2	3.6	2.0	0.5	0.5	0.4	0.4	0.3	0.3	0.3	0.4	0.4	0.4	0.5	2.0	3.5
	0.4	3.0	1.5	0.6	0.6	0.7	0.7	0.8	0.8	0.9	0.9	1.0	1.1	1.2	2.6	4.2
	0.6	2.4	1.0	1.1	1.2	1.3	1.3	1.4	1.4	1.5	1.6	1.7	1.7	1.8	3.3	4.9
	1	1.6	1.3	2.5	2.5	2.6	2.7	2.7	2.8	2.9	2.9	3.0	3.1	3.2	4.7	6.2

σ_c		ξ_z														
		-9	**-5**	**-1**	**-0.8**	**-0.6**	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**5**	**9**
	-1	62.5	88.3	77.5	65.3	44.8	44.8	44.2	45.4	43.1	37.2	41.5	85.3	46.6	23.8	26.1
	-0.6	73.2	37.1	45.2	37.8	32.9	33.0	38.8	36.2	38.6	20.1	26.8	59.4	26.5	24.1	19.1
	-0.4	41.5	43.9	37.7	24.3	49.2	51.3	22.6	42.0	21.8	29.4	41.6	28.2	26.5	18.4	11.2
	-0.2	38.2	29.2	22.6	12.2	12.2	18.7	29.5	15.1	32.9	22.2	19.8	34.5	39.6	9.3	4.3
ξ_v	**0**	27.4	23.0	11.6	15.1	13.4	9.2	18.6	16.3	11.2	15.5	10.3	19.0	9.9	6.7	1.9
	0.2	21.9	12.5	5.7	6.2	15.1	8.9	15.2	6.7	5.0	12.4	8.5	5.2	4.0	0.9	5.6
	0.4	25.4	12.7	2.2	4.7	4.7	2.3	2.6	2.4	1.8	1.9	1.7	1.6	1.7	10.9	10.4
	0.6	12.4	4.6	3.8	4.7	6.8	4.8	6.5	5.5	7.4	6.5	15.0	6.0	7.7	18.8	15.1
	1	5.9	8.6	21.2	30.1	23.6	18.1	13.8	19.9	27.6	27.7	21.2	40.6	16.7	45.1	52.1

trace the collection of outcomes under the gap rule (37a). It is seen that in both Figure 3.1- 3.2 the proposed new rule (37) dominates (37a) in two senses. First, any outcome of the rule (37a) is feasible for rule (37) and for each result of rule (37a) there are $(\sigma_\pi, \sigma_y, \sigma_c)$ which are strictly better under (37). However, there is a second sense which is also important. Rule (37) offers wider policy choice by exhibiting areas of the spaces which are feasible for a policy maker who considers all three variables $(\sigma_\pi, \sigma_y, \sigma_c)$ but not feasible under either (37a) or the Taylor Rule in Figures 2.1-2.3. Note that in Figure 3.1 the steep frontier on the left corresponds to the long curve of outcomes in Figure 3.1, reflecting Region 3 in Table 2.1. The wider frontier enables the bank to select higher or lower output or consumption volatilities than feasible under (37a) or the Taylor Rule, in exchange for inflation volatility.

Figures 3.1-3.2 PLACE HERE

(see figures on the next page)

If a central bank aims to stabilize only (σ_π, σ_y) then the gap rule is, in effect, a narrow policy which is mostly focused on the lowest left corner which is what an aggressive, inflation fighting central bank would want to attain in Region 2 of Tables 2.1-2.2. However, such a choice would imply an extremely high consumption volatility. Figure 3.2 shows that among the feasible outcomes which are efficient in the (σ_π, σ_c) space, the gap rule in favor of higher consumption volatility.

I return now to my assertion that most efficient (σ_π, σ_c) are attained by using moderate policies. Figure 4 focuses only on the proposed rule (37) and aims to clarify the contribution of aggressive ξ_π policies. To that end I select five values $\xi_\pi = 1.1, 1.3, 1.6, 1.8, 2.10$ together with $-5 \le \xi_v \le 5$, $-10 \le \xi_Z \le 10$. The (σ_π, σ_c) outcomes with these five moderate values of ξ_π are exhibited by the black circles in Figure 4.1 and trace the frontier generated by (ξ_π, ξ_v, ξ_Z). Restricting the Figure to moderate policy parameters eliminates many inefficient outcomes.

Now consider Figure 4.2 which contains all outcomes in Figure 4.1 but, in addition, the figure reports the results for the aggressive policy $\xi_\pi = 30$ which are identified with purple color at the bottom. This procedure enables us to identify outcomes under the selected values of (ξ_π, ξ_v, ξ_Z) which I view as parameters of a "moderate" policy in contrast with the aggressive anti-inflationary policy parameter $\xi_\pi = 30$. Inspection of Figure 4.2 shows that outcomes of the aggressive policy $\xi_\pi = 30$ are dominated by the moderate ξ_π policy in the usual dual sense. First, any joint policy outcome (σ_π, σ_c) attained with $\xi_\pi = 30$ can be attained with moderate policies and second, all outcomes with $\xi_\pi = 30$ result in low inflation but in very high volatility of individual consumption, in volatile bond holdings and financial markets. If we focus on the range of moderate

Figure 3.1 Policy Frontier of Output and Inflation (σ_y, σ_π) for v model

Black dots are generated by rule on state variables and inflation, red solid lines by rule on output gap and inflation. For both rules set $1.1 \leq \xi_\pi \leq 2.1$. For rule on state variables set $-10 \leq \xi_Z \leq 10$ and $-10 \leq \xi_v \leq 10$ and for rule on output gap and inflation set $-0.9 \leq \xi_y \leq 3$.

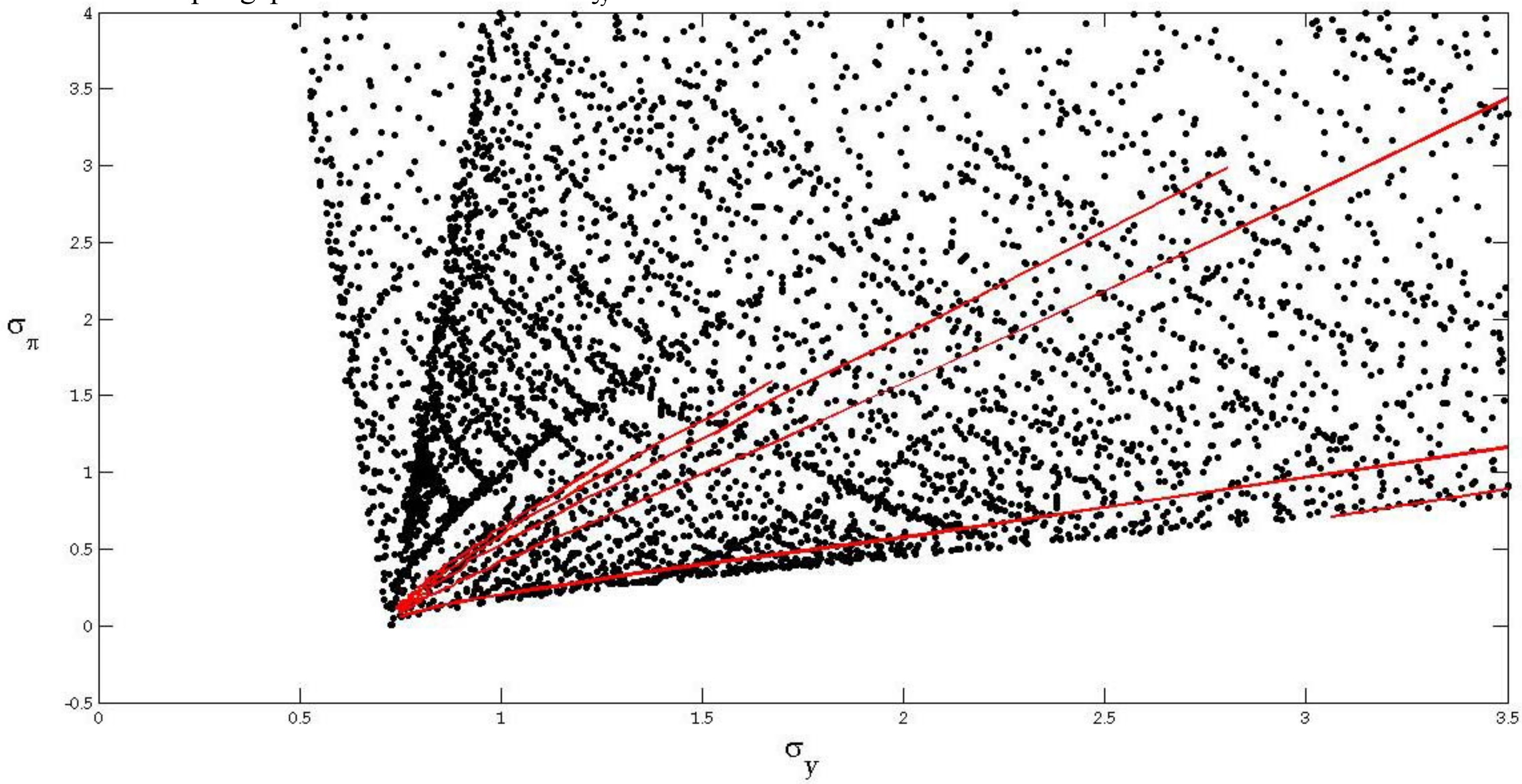
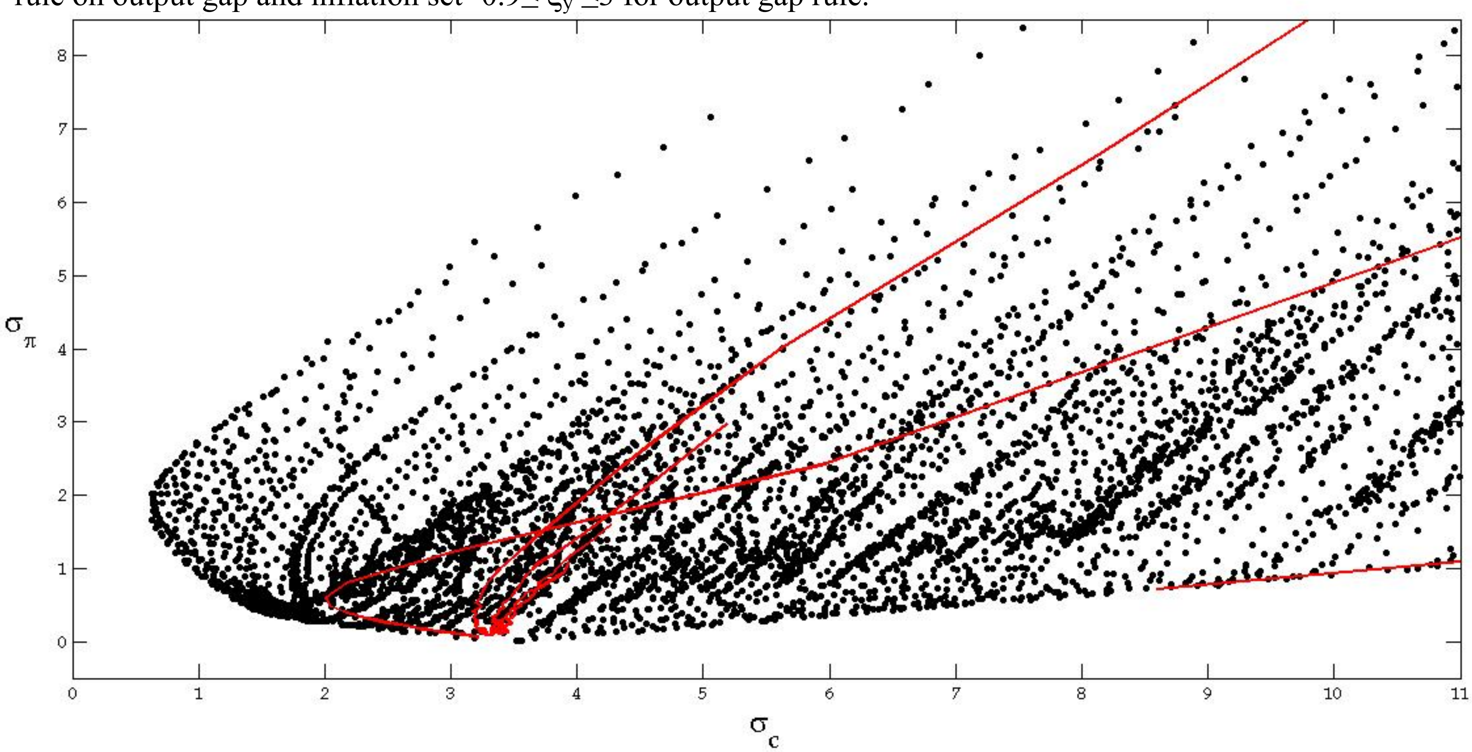


Figure 3.2 Policy Frontier of Consumption and Inflation (σ_c ,σ_π) for v model

Black dots are generated by rule on state variables and inflation, red solid lines by rule on output gap and inflation. For both rules set $1.1 \leq \xi_\pi \leq 2.1$. For rule on state variables set $-10 \leq \xi_Z \leq 10$ and $-10 \leq \xi_v \leq 10$ and for rule on output gap and inflation set $-0.9 \leq \xi_y \leq 3$ for output gap rule.



outcomes in the center of Figure 4.2 with $(\sigma_\pi \leq 2\%, \sigma_c \leq 2\%)$ it is clear that all these outcomes are generated by moderate values of the policy parameters. In short, the theory at hand explains why a central bank that is responsive to all three consequences $(\sigma_\pi, \sigma_y, \sigma_c)$ avoids aggressive anti-inflation strategies and exhibits moderation in policy choices.

Figures 4.1-4.2 PLACE HERE

(see figures on the next page)

Summary of Observations and Conclusions of the Impact of Diverse Beliefs on Policy

A short summary of the conclusions of this Section should focus on the following essential points:

- Market belief is a central sources of volatility which is more powerful than technological shocks hence a good portion of the observed $(\sigma_\pi, \sigma_y, \sigma_c)$ is caused by market expectations.
- A central bank's stabilization problem arises not only in response to sticky prices but at least as much in response to the effect of diverse market expectations. Hence, belief diversity presents a central bank with *the problem of carrying out a policy with opposition*. In order to accomplish its goals a central bank needs to adopt policies that overcome the impact on volatility of such opposition.
- The analysis shows that under diverse beliefs a policy that responds only to fluctuations in aggregate output and inflation will have non monotonic effect on volatility of output and inflation. A central bank will then need to choose between being a single mandate bank that fights only inflation and a dual mandate bank. A single mandate bank, in effect, fights to suppress the impact of expectations and such a policy will, indeed, control inflation and output volatilities but with a lower bound on σ_y and heavy cost of individual consumption volatility.
- A dual mandate central bank in Region 3 of Table 2.1-2.2 will act in a manner which is compatible with the results of Table 1.1 where a trade-off exists between inflation and output volatility.
- Either a single mandate or dual mandate central bank that chooses an aggressive ξ_π policy results in high volatility of individual consumption. Hence, under diverse beliefs a central bank objective cannot be confined to (σ_π, σ_y) but must consider the more complex set of alternative choices of $(\sigma_\pi, \sigma_y, \sigma_c)$.
- Since σ_c results from high volatility in the bond market, one must consider σ_c not only a volatility of individual consumption but also as volatility of financial markets in general.
- When a central bank is concerned with all three variables $(\sigma_\pi, \sigma_y, \sigma_c)$, an efficient policy must be moderate in nature and aggressive ξ_π anti inflationary policy is not optimal.

Figure 4.1: Most Efficient Policies are Moderate

Black dots reflect $\xi_\pi = 1.1; 1.35; 1.6; 1.85; 2.1$. Other parameters: $-5 \le \xi_v \le 5$ and $-10 \le \xi_Z \le 10$.

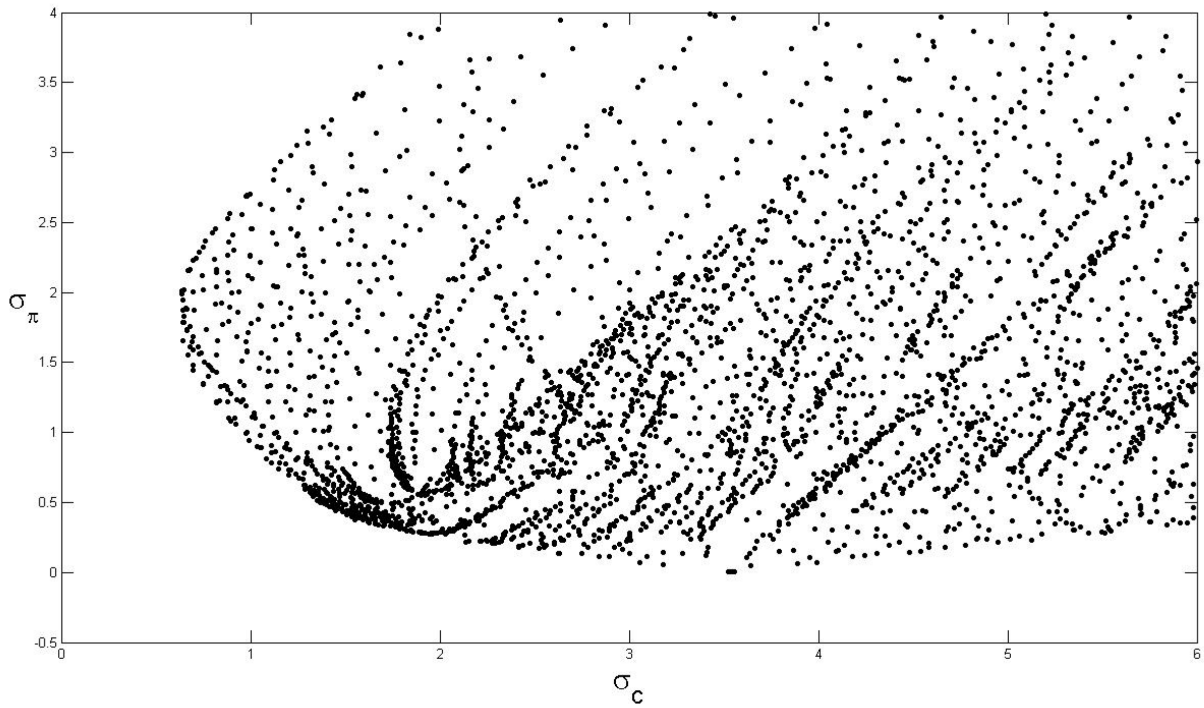
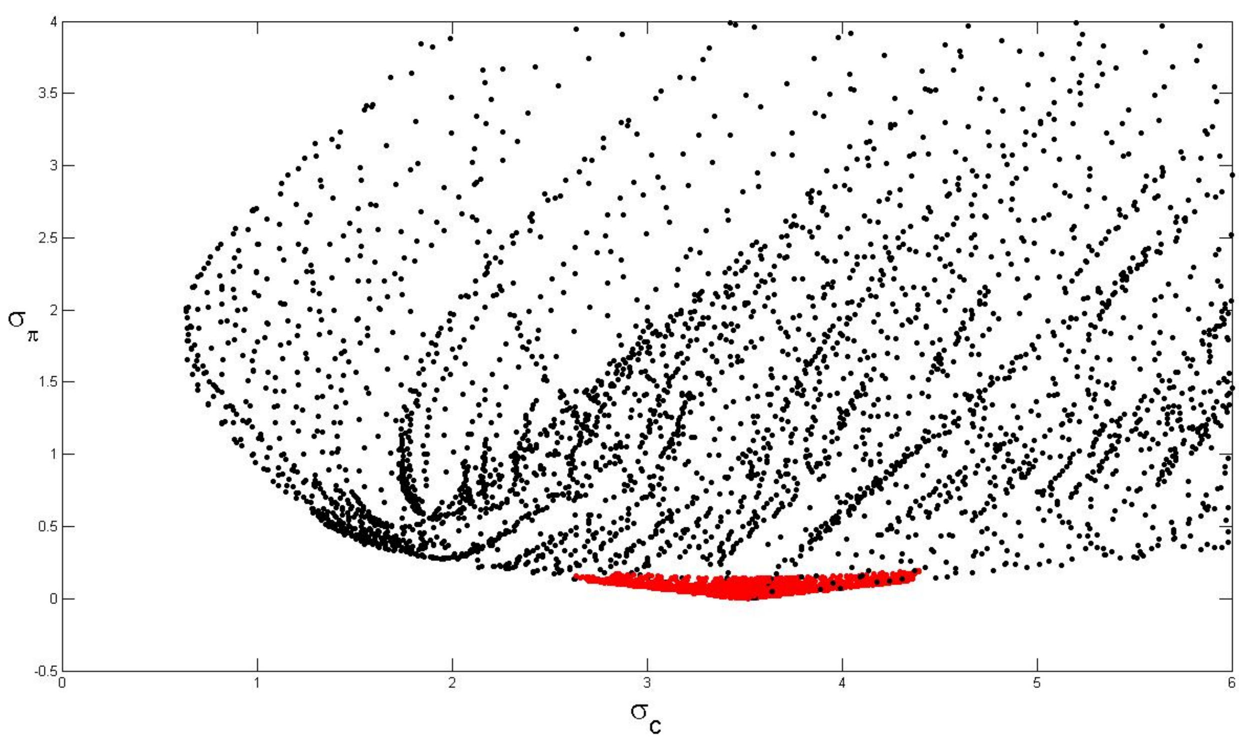


Figure 4.2: Aggressive Anti-Inflation Policies Contribute Little to the Frontier

Black dots reflect $\xi_\pi = 1.1; 1.35; 1.6; 1.85; 2.1$. Other parameters: $-5 \le \xi_v \le 5$ and $-10 \le \xi_Z \le 10$.
Red dots reflect $\xi_\pi = 30$.



7. Forward Looking Rules and Comments on Optimal Policy and Forecast Targeting

I now examine forward looking rule like $\hat{r}_t = \zeta_\pi E_t[\hat{\pi}_{t+1}] + \zeta_y E_t[\hat{y}_{t+1}]$ but for simplicity study a one shock model, excluding policy shocks u_t. With diverse beliefs a natural question arises: which expectations are to be used? Two answers come to mind. First, the central bank can use the mean market belief and employ the rule

(38a) $$\hat{r}_t = \zeta_\pi \bar{E}_t[\hat{\pi}_{t+1}] + \zeta_y \bar{E}_t[\hat{y}_{t+1}].$$

Second, the central bank may use *its own belief model* which we can denote by cb and write as

(38b) $$\hat{r}_t = \zeta_\pi E_t^{cb}[\hat{\pi}_{t+1}] + \zeta_y E_t^{cb}[\hat{y}_{t+1}].$$

These two have different implications which I explore. The key question to be resolved is how the policy rule alters the equilibrium map. In much of the monetary policy literature this issue is resolved by rewriting the *macro model* and then deducing a reduced form solution of the difference equations by forward iterations (see Blanchard and Kahn (1980)). With diverse beliefs this procedure cannot be employed without first solving two problems. First, the average market expectation operator $\bar{E}_t$ is not a conditional expectation of a proper probability and forward iterations cannot be carried out. Second one must solve the *micro equilibrium* from which to deduce missing parameters of the macro economy. Neither problem can be solved if the equilibrium state variables are not explicitly clarified and these must now be checked for the two cases noted above.

7.1 Equilibrium Under $\hat{r}_t = \zeta_\pi \bar{E}_t[\hat{\pi}_{t+1}] + \zeta_y \bar{E}_t[\hat{y}_{t+1}]$

Individual optimum conditions (3a)-(3c) and (14) take r_t as given. Hence, a perspective of dynamic optimization implies that the agent's state variables are $(Z_t, v_t, \hat{b}_{t-1}^j, g_t^j)$ as before, except for the fact that an agent must forecast interest rates. Market clearing conditions ensure that $\hat{b}_{t-1}^j$ is aggregated to zero and is not a macro state variable. Also, since in the log linearized economy the macro variables $(\hat{\pi}_t, \hat{y}_t)$ are linear in the state variables, forecasting $\hat{r}_{t+1}$ requires averaging the $(\hat{\pi}_t, \hat{y}_t)$ forecasts of others. But averaging $(Z_t, v_t, \hat{b}_{t-1}^j, g_t^j)$ yields the two state variables (Z_t, v_t) as in the case of a policy rule $\hat{r}_t = \zeta_\pi \hat{\pi}_t + \zeta_y \hat{y}_t$. An alternative and simpler argument is to assume that the micro state variables are $(Z_t, v_t, \hat{b}_{t-1}^j, g_t^j)$ and the macro state variables are (Z_t, v_t) and simply check directly for consistency. In either case we have

Proposition 8: If the policy rule is $\hat{r}_t = \zeta_\pi \bar{E}_t[\hat{\pi}_{t+1}] + \zeta_y \bar{E}_t[\hat{y}_{t+1}]$ the results of Theorem 2 remain valid: equilibrium is regular with finite memory and the policy rule can be transformed into

$$\hat{r}_t = \zeta_\pi E_t^m[\hat{\pi}_{t+1}] + \zeta_y E_t^m[\hat{y}_{t+1}] + \Gamma^r Z_t \quad , \quad \Gamma^r = \zeta_\pi \Gamma^\pi + \zeta_y \Gamma^y.$$

The conditions for determinacy are, however, different.

For two determinacy conditions that apply to this case see Galí (2008) page 79. Using the theorem above one can now rewrite the system (16a) - (16c) in the form

$$\text{(39a)} \qquad \text{IS Curve} \qquad \hat{y}_t = E_t^m(\hat{y}_{t+1}) + B_y Z_t - (\frac{1}{\sigma})[r_t - E_t^m(\hat{\pi}_{t+1})]$$

$$\text{(39b)} \qquad \text{Phillips curve} \qquad \hat{\pi}_t = \kappa(\eta + \sigma)\hat{y}_t + \beta E_t^m \hat{\pi}_{t+1} + \beta B_\pi Z_t - v_t \quad , \quad v_t \equiv \kappa(1+\eta)\hat{\zeta}_t$$

$$\text{(39c)} \qquad \text{Monetary rule} \qquad \hat{r}_t = \zeta_\pi E_t^m[\hat{\pi}_{t+1}] + \zeta_y E_t^m[\hat{y}_{t+1}] + \Gamma^r Z_t \quad , \quad \Gamma^r = \zeta_\pi \Gamma^\pi + \zeta_y \Gamma^y$$

with transitions of (v_t, Z_t). The results are easily extended to allow any other shocks.

A forward looking monetary rule has been correctly justified on sound grounds that I do not review since this paper focuses on the *impact* of diverse beliefs on the feasible outcomes of different policy rules. But a price is paid for using a rule based on forecasts. The price is an *added volatility induced by the rule itself*. In (39c) one can see it in the added term $\Gamma^r Z_t$ which amplifies volatility. It reflects uncertainty of future belief employed by the policy. It is useful to stress a general principle: *with diverse belief bank's decisions based on forecasts trigger diverse views about future beliefs employed in such forecasts and this diversity amplifies volatility*. A forward looking rule thus entails adding to that same volatility which the rule aims to stabilize! Note that a central bank can reduce its own effect on volatility by using the empirical probability m as its belief. In (39c) a credible decision by the bank will eliminate the term $\Gamma^r Z_t$.

7.2 Equilibrium Under $\hat{r}_t = \zeta_\pi E_t^{cb}[\hat{\pi}_{t+1}] + \zeta_y E_t^{cb}[\hat{y}_{t+1}]$

When a central bank uses its own forecasting model the situation changes. Understanding this helps in later discussion of "Inflation Forecast Targeting" which has received substantial attention in recent years (e.g. Svensson (2005), (2010), Svensson and Woodford (2005), Woodford (2007a), (2007b), (2010a), (2010b).

A central bank is just another agent with its own belief among rational agents and private agents do not consider the bank's belief as superior. If the bank has a credible policy in place then it does not have any information which the public does not possess since no one has private information about the macro economy. The belief of the central bank is public in the same way average private belief is public information. It is generally agreed the ability of a bank to commit to a policy is an important question and so far this paper's analysis was conducted by assuming a central bank can commit to a policy rule by developing reputation for the policy. But now, what is the confidence of the private sector in the forecast ability of the central bank? Empirical evidence suggests a central bank does not forecast inflation or GDP growth with great precision.

Table 4: Accuracy of Fed's Forecasts, Estimates of (40a)-(40b), 1965:11-1995:11
(standard errors in parentheses)

		Inflation				GDP Growth		
h	α_0^h	α_1^h	R^2	N	δ_0^h	δ_1^h	R^2	N
0	.35 (.22)	.97 (.04)	.83	294	.80 (.30)	.89 (.08)	.53	293
1	.36 (.31)	1.00 (.07)	.72	278	.88 (.54)	.77 (.13)	.25	277
2	.32 (.36)	1.03 (.08)	.60	256	.70 (.67)	.79 (.18)	.18	255
3	.29 (.38)	1.04 (.08)	.54	239	1.05 (.93)	.62 (.27)	.08	238
4	- .13 (.41)	1.09 (.09)	.54	209	- .15 (1.07)	1.08 (.33)	.16	208
5	- .36(.42)	1.08 (.10)	.54	250	- .70 (1.15)	1.31 (.41)	.18	149
6	- .25(.45)	.95 (.14)	.59	90	- .80 (1.11)	1.47 (.41)	.22	89
7	- .09(.56)	82 (.19)	.64	59	- .26 (1.93)	1.35 (.71)	.10	58

Table 4 reports on the Fed's staff forecast accuracy during the period 1965:11-1995:11. $x_{t,h}$ is actual value h quarters after the forecast date t and the estimated equations for horizons of h = 0, 1, 2, ..., 7 quarters (0 means present quarter since data is not released until the end of the quarter) are:

(40a) $$\pi_{t,h} = \alpha_0^h + \alpha_1^h \pi_{t,h}^e + \varepsilon_{t,h}$$

(40b) $$g_{t,h} = \delta_0^h + \delta_1^h g_{t,h}^e + \vartheta_{t,h}.$$

N is the number of observations and R^2 is adjusted. Note that although time series of inflation are very persistent, the Fed's forecasts explain only slightly more than half of actual variability of inflation. The Fed's forecast accuracy of GDP growth for h > 0 is very poor and for h > 1 is virtually useless since with such low R^2 and high standard errors, these forecasts are very unreliable.

Consider Table 4 in relation to the literature on optimal policy and implementation of optimal policy with forecast targeting. A low forecast accuracy of the components used to forecast optimal interest rates mean that if forecast targeting is implemented and if forecasts are made with an inflation and output targets over a span like 12 quarters, these forecasts will materialize with very low probability and convergence to the targets within that time is very unlikely. One must then view forecast targeting not as a definite course of action to be undertaken by the central bank but rather as an expression of what the bank aims to accomplish and its formal strategy of attaining these stabilization goals. Some of the writings on forecast-targeting suggest that a central bank needs to"convince" the public or "shape" expectations of the private sector. The idea of managing public expectations is questionable and I will show it is not necessary within the RE framework of modeling the implementation of optimal policy via forecast targeting. Let me explain.

The "Inflation-Forecast Targeting" literature universally assumes *homogenous beliefs* in the form of RE . Hence, for a given policy (i.e. a state contingent plan), the central bank and the private sector have *the same conditional expectations* and as long as the public believes the policy is being carried out there is no disagreement between private agents and the central bank on any forecasts. In an economy with a single belief, if there is agreement on policy, the bank does not need to convince anyone or shape anyone's expectations. Actually, a central bank needs only specify its objective function and both the public and the bank can compute the same optimal inflation and growth paths as well as the conditional path of interest rates the bank will set given the states. Yet, the *words* used in the discussion of forecast targeting literature imply some vague disagreement between the central bank and the private sector which requires the bank to influence or shape private expectations. Consider, for example, the following statement in Woodford (2007b):

> This approach has important advantages as a way of *shaping private-sector expectations*. On the one hand, a commitment to regular publication of a detailed analysis that shows how specific policy decisions conform to a general decision framework makes it evident to the public that it can count on the bank to conduct policy in a specific, relatively predictable way. Moreover, the emphasis on the bank's projections of the economy's evolution directs attention very precisely to the implications of the policy framework for *expectations that the central bank would like the public to share...* (my italics)

Similar statements are found in other papers. Two additional facts about Fed forecasting need to be mentioned. First, the forecasts used to estimate the equations in Table 4 are made by the *Federal Reserve staff* and are released five years later. If the aim is to inform the public about the Fed's views why are the forecasts released only five years later? Second, members of the open market committee make their own individual forecasts and these are released with the committee's minutes. Examination of these reveal wide differences in forecasts of GDP growth and inflation among members even for relatively short horizons. With wide differences of forecasts *within* the Fed, any official "Central Bank Forecast" needed for "forecast targeting policy" can only be some sort of a compromise the nature of which will only trigger market speculations.

I stress that my aim here is not to criticize the literature on forecast targeting. The issue with which I am concerned arises from the fact that there is a deep difference between an optimal monetary policy and forecast targeting in an economy with a single belief and a policy in an economy with diverse beliefs. In a world with diverse beliefs forecast targeting requires the central bank to take into account the fact that the bank's forecasts may have its own contribution to that same volatility it wishes to control. The difference between the bank and private agents *may not necessarily be about the bank's intent or ability to carry out a consistent policy*. Even with full central bank credibility under which the private sector fully accepts the bank's policy commitment, differences will exist between the bank's and the private sector's forecasts of future factors that affect policy. The bank and private agents may hold different forecasts of exogenous shocks and, most important, *of future states of belief of the market and the bank itself*. It is shown that future states of belief are key components of an optimal policy and differences in forecasting them impact the policy. I examine below if one can bridge these differences. With this in mind I suggest the literature on forecast targeting actually recognizes that a conflict between the central bank's belief and private sector beliefs create real problems. While the words used in that literature reflect the recognition of these difficulties, the RE based models used in this literature do not present any such difficulties or conflicts. I explore this issue further after examining how the model is altered by the policy rule (38b).

To model the central bank as an agent requires us to specify the belief index g_t^{cb} of the central bank. Using the same logic as the private sector, one establishes the transition of g_t^{cb} to be as in (25c) and express the belief model of the bank by $(v_{t+1}^{cb}, Z_{t+1}^{cb}, g_{t+1}^{cb})$ with transitions

(41a) $$v_{t+1}^{cb} = \lambda_v v_t + \lambda_v^{gcb} g_t^{cb} + \rho_{t+1}^{cbv}$$

(41b) $$Z_{t+1}^{cb} = \lambda_Z Z_t + \lambda_Z^v [v_{t+1} - \lambda_v v_t] + \lambda_Z^{gcb} g_t^{cb} + \rho_{t+1}^{cbZ}$$

(41c) $$g_{t+1}^{cb} = \lambda_Z g_t^{cb} + \lambda_Z^v [v_{t+1} - \lambda_v v_t] + \rho_{t+1}^{cbg}$$

$$\begin{pmatrix} \rho_{t+1}^{cbv} \\ \rho_{t+1}^{cbZ} \\ \rho_{t+1}^{cbg} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0, \\ 0 \end{matrix} \begin{bmatrix} \hat{\sigma}_v^2, & 0, & 0 \\ 0, & \hat{\sigma}_Z^2, & \hat{\sigma}_{Zg}, \\ 0, & \hat{\sigma}_{Zg}^2, & \sigma_g \end{bmatrix} \right)$$

and a normalization $\lambda_v^{gcb} = 1$. (41c) is the empirical distribution of g_t^{cb}. Agents take g_t^{cb} as a new and given state variable hence agent j's vector of state variables becomes $(Z_t, v_t, \hat{b}_{t-1}^j, g_t^j, g_t^{cb})$. Naturally, the central bank may adopt other forms of belief. But one must recognize the impact of the new state variable g_t^{cb} which triggers private sector speculations about its future evolution. One immediate question is whether (41c) is the true transition of g_t^{cb}? Any new relevant state variable triggers private speculations which complicates the market's belief. In this case private belief changes from (25a)-(25c) to a general perception by i

(42a) $$v_{t+1}^i = \lambda_v v_t + \lambda_v^g g_t^i + \rho_{t+1}^{iv}$$

(42b) $$Z_{t+1}^i = \lambda_Z Z_t + \lambda_Z^v [v_{t+1} - \lambda_v v_t] + \lambda_Z^g g_t^i + \rho_{t+1}^{iZ}$$

(42c) $$g_{t+1}^{cbi} = \lambda_Z g_t^{cb} + \lambda_Z^v [v_{t+1} - \lambda_v v_t] + \lambda_{cb}^g g_t^i + \rho_{t+1}^{cbg}$$

(42d) $$g_{t+1}^i = \lambda_Z g_t^i + \lambda_Z^v [v_{t+1} - \lambda_v v_t] + \rho_{t+1}^{ig}$$

with a covariance matrix. In (42a)-(42d) a belief state g_t^i impacts three perceived transitions of macro state variables (v_t, Z_t, g_t^{cb}) taken exogenously by i. Both private agents and the central bank formulate belief about future business conditions expressed by values of v_{t+1}. This triggers an expanded individual state space and belief about future market belief Z_{t+1} and future central bank belief g_{t+1}^{cb} with parameters $(\lambda_Z^g, \lambda_{cb}^g)$. Note the general principle implied: an ambiguity about the future leads to an expansion of the issues subject to diverse belief and further amplification of market volatility. By basing policy on its own belief, a central bank opens the door for the market to endogenously add a component of uncertainty which was not there before.

Aggregation and market clearing conditions show that macro state variables are (Z_t, v_t, g_t^{cb}). In this case an equilibrium with the central bank as an agent becomes *regular with finite memory* hence using (41a) - (41c) one can compute the new policy rule as follows. First compute the differences

$$E_t^{cb}[\hat{\pi}_{t+1}] - E_t^{m}[\hat{\pi}_{t+1}] = [(A_\pi^Z + A_\pi^g)\lambda_Z^v + A_\pi^v + A_\pi^{gcb}\lambda_Z^{gcb}]g_t^{cb}$$
$$E_t^{cb}[\hat{y}_{t+1}] - E_t^{m}[\hat{y}_{t+1}] = [(A_y^Z + A_y^g)\lambda_Z^v + A_y^v + A_y^{gcb}\lambda_Z^{gcb}]g_t^{cb}.$$

Next, write the new policy rule for the linearized economy expressed in terms of the probability m

$$\hat{r}_t = \zeta_\pi E_t^{cb}[\hat{\pi}_{t+1}] + \zeta_y E_t^{cb}[\hat{y}_{t+1}] = \zeta_\pi E_t^{m}[\hat{\pi}_{t+1}] + \zeta_y E_t^{m}[\hat{y}_{t+1}] + [\zeta_\pi \Gamma^{\pi cb} + \zeta_y \Gamma^{ycb}]g_t^{cb} \tag{43}$$

where $\Gamma^{\pi cb} = [(A_\pi^Z + A_\pi^g)\lambda_Z^v + A_\pi^v + A_\pi^{gcb}\lambda_Z^{gcb}]$, $\Gamma^{ycb} = [(A_y^Z + A_y^g)\lambda_Z^v + A_y^v + A_y^{gcb}\lambda_Z^{gcb}]$.

But recall that the equilibrium parameters $(A_\pi^Z, A_\pi^g, A_\pi^v, A_\pi^{gcb}, A_y^Z, A_y^g, A_y^v, A_y^{gcb})$ *depend upon the policy!*

The conclusion I draw from (39a)-(39c) and (43) is that, in the case at hand, in which the central bank's belief is Markov and the empirical probability m is Markov, the resulting macro economic equilibrium has the same analytical structure as (30a)-(30c). Different policies will surely exhibit drastically different dynamic properties but the basic causal structure remains the same: market belief *does not have an effect on determinacy* and Propositions 3 and 4 continue to hold. Since this macro system has different parameters the conditions for determinacy are different but employ the same formula (see Galí (2008), page79).

7.3 On Optimal Monetary Policy and Forecast Targeting in an Economy with Diverse Beliefs

I turn now to the standard example of quadratic objective function where optimal monetary policy minimizes $E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\frac{1}{2}[\hat{\pi}_t^2 + \Xi(\hat{y}_t - \hat{y}^*)^2]$ subject to a Phillips curve where $\hat{y}^*$ is a target output deviation. It leads to an optimal sequence of state contingent functions $(\hat{\pi}_t, \hat{y}_t)$. These are inserted into the IS curve to deduce an optimal interest rate function selected by the central bank to implement the optimal sequence[10]. A policy is now defined as a state contingent interest rate function which aims to attain a specific optimal time path of $(\hat{\pi}_t, (\hat{y}_t - \hat{y}^*))$. Can we carry out a similar program for an economy with diverse beliefs?

One question faced at the outset is obvious: which probability should be used for the objective? The bank cannot use the mean belief operator $\bar{E}_t$ since it is not derived from a proper probability. The option available is to use the central bank's own belief and a formulation of it is offered in (41a)-(41c). It would be different from the beliefs of the private sector. More difficult is the question of the Phillips Curve constraints for the optimization. Without any knowledge about the equilibrium, all we could do is use the formulation

[10] Determinacy of the resulting competitive equilibrium (see Sargent and Wallace (1975) is assured by the fact that an optimal interest rate policy is a conditional plan expressed by a sequence of forecasts of endogenous variables, implemented by the central bank via a mechanism discussed below.

of the *untransformed* Phillips curve in (16b) defined by

$$\hat{\pi}_t = \kappa(\eta + \sigma)\hat{y}_t + \beta \overline{E}_t \hat{\pi}_{t+1} + \beta(1-\omega)\Phi_t(\hat{q}) + v_t .$$

Such constraint presents a key analytical problem. It deserves a restatement in more general terms: *A central bank's objective is to optimize by using an expectation* $E_{t_0}^{cb}$ *deduced from its own belief, subject to the Phillips curve which is a behavioral relation defined in terms of diverse private beliefs and optimal choices.* This basic problem will be present regardless of the specific formulation of individual beliefs and the discussion here is an exploration of the problem as it is manifested in the model at hand.

A minimization subject to the Phillips curve (16b) is a complex central bank optimum of selecting a $(\hat{\pi}_t, \hat{y}_t)$ sequence. Such objective uses a bank's probability but (16b) is defined by average expectations relative to an infinite number of other probabilities and with the term $\Phi_t(\hat{q}) = \int_0^1 \left(E_t^j \hat{q}^{\star}_{j(t+1)} - E_t^j \hat{q}_{(t+1)}\right) dj$. Hence, a bank needs to know how individual beliefs and choices vary with $(\hat{\pi}_t, \hat{y}_t)$. One possible solution is analogous to Theorem 2 and (42) and aims to permit an equilibrium transformation of the central bank's expectations as well as the Phillips and the IS curves. Such solution is possible when we know that for a given policy the resulting competitive equilibrium is *regular*. But this is a circular condition: constructing an optimal policy depends on the regularity of equilibrium while regularity cannot be established without knowledge of the optimal policy. To do that we need to establish conditions to ensure an invariant policy (i.e. not time dependent) and a resulting equilibrium which is regular. But then, assuming the equilibrium is regular, what difficulties does this procedure present?

To explore the issues I simply *assume* an equilibrium ends up regular and decisions are functions of finite number of state variables. With this assumption one deduces the optimal policy and proves later the constructed policy is consistent with the assumed regularity of competitive equilibrium. I briefly carry out this procedure by stating the optimization problem:

$$\text{Minimize } E_{t_0}^{cb} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{1}{2} [\hat{\pi}_t^2 + \Xi(\hat{y}_t - \hat{y}^*)^2]$$

subject to (see (30b))

1. $\hat{\pi}_t = \kappa(\eta + \sigma)\hat{y}_t + \beta E_t^m \hat{\pi}_{t+1} + \beta B_\pi Z_t + v_t .$
2. E_t^{cb} defined by (41a)-(41c).

The term $\beta B_\pi Z_t$ *depends upon the parameters of the micro competitive equilibrium*. In the minimization below I ignore this dependence of B_π on the policy and treat it as a constant, but keep it in mind.

From (41a)-(41c) it follows that for any macro variable x which is a linear function of the states, $E_t^{cb}[x_{t+1}] = E_t^m[x_{t+1}] + B_x^{cv} g_t^{cb}$ for some constant B_x^{cv} to be determined in equilibrium. Denoting by φ_t the multiplier, I write the Lagrangian

(44) $$\mathcal{L}_{t_0} = E_{t_0}^{cb} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{1}{2}[\hat{\pi}_t^2 + \Xi(\hat{y}_t - \hat{y}^*)^2] + \varphi_t[\hat{\pi}_t - \kappa(\eta + \sigma)\hat{y}_t - \beta E_t^m \hat{\pi}_{t+1} - \beta B_\pi Z_t - v_t].$$

But given the regularity of equilibrium I transform the bank's expectations

$$E_t^{cb}\hat{\pi}_{t+1} = E_t^m \hat{\pi}_{t+1} + B_\pi^{cb} g_t^{cb}$$

and rewrite (44) as

$$\mathcal{L}_{t_0} = E_{t_0}^{cb} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{1}{2}[\hat{\pi}_t^2 + \Xi(\hat{y}_t - \hat{y}^*)^2] + \varphi_t[\hat{\pi}_t - \kappa(\eta + \sigma)\hat{y}_t - \beta(E_t^{cb}\hat{\pi}_{t+1} - B_\pi^{cb} g_t^{cb}) - \beta B_\pi Z_t - v_t].$$

Use the law of iterated expectations to deduce

(45) $$\mathcal{L}_{t_0} = E_{t_0}^{cb} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \frac{1}{2}[\hat{\pi}_t^2 + \Xi(\hat{y}_t - \hat{y}^*)^2] + \varphi_t[\hat{\pi}_t - \kappa(\eta + \sigma)\hat{y}_t - \beta\,\hat{\pi}_{t+1} + (\beta B_\pi^{cb} g_t^{cb} - \beta B_\pi Z_t - v_t)].$$

Optimum requires

(45a) $$\hat{\pi}_t + \varphi_t - \varphi_{t-1} = 0$$

(45b) $$\Xi(\hat{y}_t - \hat{y}^*) - \kappa(\eta + \sigma)\varphi_t = 0$$

and a technical condition to ensure stationarity by requiring independence of t_0. Conditions (45a)-(45b) are identically the same as under homogenous belief (e.g. conditions (1.9)-(1.10) in Woodford (2010b)) . But there is a difference. To solve for φ_t compute the values of $(\hat{\pi}_t, \hat{y}_t - \hat{y}^*)$ in (45a)-(45b), insert into the transformed Phillips curve defined by

(45c) $$\hat{\pi}_t = \kappa(\eta + \sigma)\hat{y}_t + \beta E_t^{cb}\hat{\pi}_{t+1} - \beta B_\pi^{cb} g_t^{cb} + \beta B_\pi Z_t + v_t$$

and deduce a second order difference equation of φ_t in $(-\beta B_\pi^{cb} g_t^{cb} + \beta B_\pi Z_t + v_t)$. Then, given a solution of φ_t, the solution of $\hat{\pi}_t$ is deduced from (45a), the solution of $\hat{y}_t - \hat{y}^*$ from (45b) and the solution of the optimal interest rate function from the transformed IS curve. The two key difficulties are now clear.

(i) An optimal policy depends on the bank's forecasted forward path of $(-\beta B_\pi^{cb} g_t^{cb} + \beta B_\pi Z_t + v_t)$, but these forecasts and their parameters depend on the policy. Such a fixed point between policy and equilibrium is central and not new. The key point here is that it arises from the fact that agents have diverse beliefs.

(ii) Even if the bank takes a neutral stance and adopts the empirical probability m as its belief (thus eliminating the term $\beta B_\pi^{cb} g_t^{cb}$), it must still acknowledge its belief is different from the public's belief and for an optimal policy it must forecast future mean market belief. Moreover, *the optimal interest policy depends upon the mean market belief as well as the bank's own belief.*

These two problems explain that under diverse beliefs the nature of an optimal policy is different from a policy in an RE based model, and that such an optimal policy is difficult for a bank to implement. First, the bank is not a central planning bureau that computes fixed points to attain consistency between policy and equilibrium. Second, in forecasting the market belief and its own belief, two steps needed for a computation of φ_t, the bank needs to adapt its political outlook on stabilization policy. Why?

To start with, one may argue the bank can forecast future Z_t and g_t^{cb} *privately* and make public only forecasts of inflation and the output deviation since (45a)-(45b) imply

(46) $$\hat{\pi}_t + \Psi(\hat{y}_t - \hat{y}_{t-1}) = 0 \qquad , \qquad \Psi = \frac{\Xi}{\kappa(\eta + \sigma)} .$$

Indeed, (46) is the basis for making sequential forecasts used for "forecast targeting" of the policy in place. This requires an explicit statement by the bank that for all j > 0

(47) $$E_t^{cb}\hat{\pi}_{t+j} + \Psi(E_t^{cb}\hat{y}_{t+j} - E_t^{cb}\hat{y}_{t+j-1}) = 0$$

and that the policy is compatible with equilibrium (i.e. the bank solved the fixed point problem). The argument supporting "forecast targeting" says a central bank needs only carry out the sequential forecasts as in (46)-(47). The reasons being (i) that such simple sequential forecasting is *equivalent* to an interest rate policy deduced from the IS curve, and (ii) that the public, who needs to know the interest rate policy since consumption, savings and security purchases are made based on interest rate forecasts, can make this deduction on its own, from the forecasts. But here this argument means that to deduce an optimal interest rate the bank must solve the joint system of (47) and the transformed Phillips curve (45c) which depends upon $(-\beta B_\pi^{cb} g_t^{cb} + \beta B_\pi Z_t + v_t)$. We are then back to the same problem: to deduce an optimal interest rate policy the bank must forecast (g_t^{cb}, Z_t, v_t). Moreover, since agents' beliefs are different from the bank's, for the public to understand the interest rate policy the public must (i) know the bank's belief dynamics, (ii) use the transformed Phillips and IS curves which requires equilibrium $(B_\pi, B_y, B_\pi^{cb}, B_y^{cb})$, and (iii) use the forecasted forward paths of the three variables (g_t^{cb}, Z_t, v_t) needed to solve (47). Clearly, the public cannot deduce the optimal policy just from a statement of the forecasts as in (47).

To see the policy implications of the above results I start with the *narrow and direct* implications.

(i) Without further simplifications the bank will find it technically difficult to realize an optimal policy since it requires a solution of a complex fixed point problem to make policy and equilibrium compatible.

(ii) An optimal policy entails actions a central bank must take when it disagrees with the market's belief and a transparent policy is explicit and open about this disagreement. Hence, forecast targeting of inflation and output deviation only are not sufficient since the bank cannot change the belief of the private sector about future paths of state variables. An optimal policy may need, therefore, to suppress the impact of the market's belief on economic fluctuations by altering the private sector's cost of acting upon those beliefs.

(iii) The conclusion that an optimal policy responds to market belief supports the earlier conclusion about the gain from including the mean market belief Z as an explicit variable in any fixed policy rule.

(iv) A central bank's optimal policy is not optimal relative to utilities and beliefs of individual agents who, therefore, oppose the policy. Such differences are not as small as in Woodford (2010a). The realism of this conclusion is demonstrated by testimonies in Congressional records and statements by experts, including academic scholars and Wall Street's analysts, who express wide opposition to Fed policy.

I turn now to the *broader implications* of an optimal policy in an economy with diverse beliefs. Recall first a general result about competitive equilibria of such economies which states that competitive allocations of such economies are not Pareto Optimal (see Hammond (1983), Nielsen (2003), (2011)) even if markets are complete. The reason is that if the added risk induced by beliefs is removed, allocations exist which are less risky for all. The result is reinforced by the fact that markets are inherently incomplete since markets for claims which are contingent on market belief cannot exist for natural reasons. The implication is that an optimal monetary policy is bank's optimal among feasible competitive programs but is not Pareto Optimal and may not even be Pareto improving relative to an initial fixed monetary rule. It is entirely possible for an opponent of an optimal monetary policy to rationally support a fixed monetary rule as a superior alternative policy. But then the implications of this discussion are clear.

(i) Since an optimal policy is not Pareto improving the perspective of a central bank's problem offered here is different from the one under consensus. In contrast with the view that a central bank either reflects the public's beliefs or that it "persuades" the public to accept the bank's forecasts, the view offered here is one of a central bank that reduces social risk (i.e. volatility) by changing the equilibrium map but who cannot change individual beliefs about state variables. The bank and the private sector may thus disagree about forecasts of endogenous variables and policy is conducted with opposition.

(ii) Although the financial structure of the model here is simple, in real markets an optimal policy needs to respond to manifestations of market belief such as rapidly changing asset prices, often called "bubbles" in the popular press, or low levels of investment due to excessively low business "confidence." As with fixed policy rules, an efficient way to respond to market belief is for bank policy to respond to asset prices.

(iii) An optimal policy relative to a bank's belief adds to privately perceived risk defined by future bank's belief. This gives rise to a desire for market trading to hedge this risk. One of the reasons for market trading in Federal Funds futures is the risk of what the bank's future belief will be.

(iii) Aiming for Pareto improving policy, a central bank must select its objective and belief so as to have as much public support as possible and minimize the added risk generated by policy. This suggests the central bank selects a median voter objective and use the empirical stationary probability m as its belief. In doing so the bank needs to explain exactly how forecasts are made. If the public is convinced the forecasts are made without judgment, it will eliminate the term $\beta B_{\pi}^{cb} g_t^{cb}$ from perceived private risk.

There is one conclusion that emerges from examining fixed monetary rules and optimal monetary policy. The theory developed here shows that either policy must respond to market belief. Although an optimal policy in the sense defined above may not have a "good" outcome, a central bank should certainly aim for *Pareto improving policy* hence implementing such policy is a correct goal. However, a central bank must select an objective which enables it to execute a desirable policy and convey it to the public in a simple and transparent manner with a view to attain maximal public support and entail minimum additional public risk generated by the policy itself. Conclusions on these issues deduced from a model based on a single homogenous belief need to be seriously questioned.

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APPENDIX A: Proof of Theorem 1

Assumption 3: The alternate prior, based on the public signal, incorporates a direct learning process where

$$S_t^{\ i} \sim N(\Psi_t^i, \frac{1}{\gamma}).$$

One interpret Ψ_t^i as a prior subjective mean, positive or negative, *given the qualitative public signal.*

<u>Proof</u> : Using Assumption 3 combine the two sources to have that

$$s_t(v_t, \Psi_t^i) = \mu s_{t-1}(v_t) + (1-\mu) S_t^{\ i}$$

with a mean of

$$E_t^{\ i}(s_t | v_t, \Psi_t^i) = \mu E_t^{\ i}(s_{t-1} | v_t) + (1-\mu)\Psi_t^i \qquad 0 < \mu < 1$$

and conditional variance

$$\mathbf{Var}(s_t | v_t, \Psi_t^i) = \frac{\mu^2}{(\alpha + \nu)} + \frac{(1-\mu)^2}{\gamma}.$$

Let $\zeta = \frac{1}{\mu^2}$ **and** $\xi = \frac{1}{(1-\mu)^2}$ and we write the precision of the distribution of this new posterior as

$$\mathbf{Precision}(s_t | v_t, \Psi_t^i) = \Gamma(s_t) = 1/[\frac{1}{\zeta(\alpha + \nu)} + \frac{1}{\xi\gamma}] = \frac{\zeta(\alpha + \nu)\xi\gamma}{\zeta(\alpha + \nu) + \xi\gamma}$$

At date t+1 the agent observes v_{t+1}. By (18a) in the text in the form $v_{t+1} - \lambda_v v_t = s_t + \varepsilon_{t+1}^v$, $\varepsilon_{t+1}^v \sim N(0, \frac{1}{\nu})$ it follows that updating $E_t^{\ i}(s_t | d_t, \Psi_t^i)$ the agent has

$$E_{t+1}^{\ i}(s_t | v_{t+1}, \Psi_t^i) = \frac{\Gamma(s_t) E_t^{\ i}(s_t | v_t, \Psi_t^i) + \nu[v_{t+1} - \lambda_v v_t]}{\Gamma(s_t) + \nu} \quad , \quad s_t(v_{t+1}, \Psi_t^i) \sim N[E_t^{\ i}(s_t | v_{t+1}, \Psi_t^i), \frac{1}{\Gamma(s_t) + \nu}].$$

After assessing the mean Ψ_{t+1}^i he formulates the new posterior which is

$$s_{t+1}(v_{t+1}, \Psi_{t+1}^i) = \mu s_t(v_{t+1}, \Psi_t^i) + (1-\mu) S_{t+1}^{\ i}$$

with mean

(A.1) $$E_{t+1}^{\ i}(s_{t+1} | v_{t+1}, \Psi_{t+1}^i) = \mu E_{t+1}^{\ i}(s_t | v_{t+1}, \Psi_t^i) + (1-\mu)\Psi_{t+1}^i \qquad 0 < \mu < 1.$$

conditional variance

(A.2) $$\mathbf{Var}(s_{t+1} | v_{t+1}, \Psi_{t+1}^i) = \frac{1}{\zeta(\Gamma(s_t) + \nu)} + \frac{1}{\xi\gamma}.$$

and precision

(A.3) $$\Gamma(s_{t+1}) = 1/[\frac{1}{\zeta(\Gamma(s_t) + \nu)} + \frac{1}{\xi\gamma}] = \frac{\zeta(\Gamma(s_t) + \nu)\xi\gamma}{\zeta(\Gamma(s_t) + \nu) + \xi\gamma}.$$

We can now deduce the full symmetry of the process. For large t we then have

$$s_t(v_{t+1}, \Psi_t^i) \sim N[E_t^{\ i}(s_t | v_{t+1}, \Psi_t^i), \frac{1}{\Gamma(s_t) + \nu}]$$

$$E_t^i(s_t|v_{t+1},\Psi_t^i) = \frac{\Gamma(s_t)E_t^i(s_t|v_t,\Psi_t^i) + \nu[v_{t+1} - \lambda_v v_t]}{\Gamma(s_t) + \nu}.$$

After observing Ψ^i_{t+1} the new posterior is

$$s_{t+1}(v_{t+1},\Psi^i_{t+1}) = \mu s_t(v_{t+1},\Psi_t^i) + (1-\mu)S^i_{t+1}.$$

The mean, conditional variance and precision are then as in (A1), (A.2) and (A.3) and hence we have an equation for the precision

$$\Gamma_{t+1} = \frac{\zeta(\Gamma_t + \nu)\xi\gamma}{\zeta(\Gamma_t + \nu) + \xi\gamma}.$$

It is well defined for $1 < \zeta < \infty$ (i.e. $0 < \mu < 1$) and in that case it has the unique positive solution

$$\zeta = \frac{1}{\mu^2} \text{ and } \xi = \frac{1}{(1-\mu)^2} \quad , \qquad \Gamma^* = \frac{(\nu + \xi\gamma(1-\frac{1}{\zeta})) + \sqrt{(\nu + \xi\gamma(1-\frac{1}{\zeta}))^2 + 4\nu}}{2} > 0.$$

The negative root has no economic meaning. When $\zeta = 1$, $\xi = \infty$ there is no solution, and Γ_t diverges for large t, which is the classical case. Now insert $\Gamma = \Gamma^*$ into the the equations above to deduce that

$$E^i_{t+1}(s_{t+1}|v_{t+1},\Psi^i_{t+1}) = \mu E^i_{t+1}(s_t|v_{t+1},\Psi_t) + (1-\mu)\Psi^i_{t+1}$$

hence

(A.4) $$E^i_{t+1}(s_{t+1}|v_{t+1},\Psi^i_{t+1}) = \mu\frac{\Gamma^* E_t^i(s_t|v_t,\Psi_t^i) + \nu[v_{t+1} - \lambda_v v_t]}{\Gamma^* + \nu} + (1-\mu)\Psi^i_{t+1}.$$

Now define

$$g^i_{t+1} = E^i_{t+1}(s_{t+1}|v_{t+1},\Psi^i_{t+1}) \quad , \quad \lambda_Z = \frac{\mu\Gamma^*}{\Gamma^* + \nu} > 0 \quad , \quad \lambda_Z^v = \frac{\mu\nu}{\Gamma^* + \nu} > 0, \quad , \quad \rho^{ig}_{t+1} = (1-\mu)\Psi^i_{t+1}.$$

Hence, the law of motion of g^i_{t+1} is

(A.5) $$g^i_{t+1} = \lambda_Z g_t^i + \lambda_Z^v[v_{t+1} - \lambda_v v_t] + \rho^{ig}_{t+1}.$$ ■

Some Comments.

Aggregation implies an empirical distribution of the form

(A.6) $$Z_{t+1} = \lambda_Z Z_t + \lambda_Z^v[v_{t+1} - \lambda_v v_t] + \rho^Z_{t+1}$$

and a belief of i in the mean belief of others is defined by

$$Z^i_{t+1} = \lambda_Z Z_t + \lambda_Z^v[v_{t+1} - \lambda_v v_t] + \lambda_Z^g g_t^i + \rho^{iZ}_{t+1}.$$

For simulations one uses the expressions (A.5) and (A.6). General equilibrium computations are based on expectations of (A.5) - (A.6) which are:

$$E_t^i(g^i_{t+1}) = (\lambda_Z + \lambda_Z^v)g_t^i$$
$$E_t^i(Z^i_{t+1}) = \lambda_Z Z_t + (\lambda_Z^v + \lambda_Z^g)g_t^i.$$

These are then used in the general equilibrium computations of (A_y, A_π). In addition, we have that

(A.7) $$\int_0^1 [E_t^i(g^i_{t+1}) - E_t^i(Z^i_{t+1})]\,dt = \lambda_Z^g Z_t.$$

APPENDIX B: Identification of parameters with u = 0 .

Decision functions in the log-linearized economy take the following form:

(B.1a) $\hat{c}_t^{\,j} = A_y^{\;Z} Z_t + A_y^{\;v} v_t + A_y^{\;b} \hat{b}_{t-1}^{\,j} + A_y^{\;g} g_t^{\,j} = A_y \bullet (Z_t, v_t, \hat{b}_{t-1}^{\,j}, g_t^{\,j})$

(B.1b) $\hat{b}_t^{\,j} = A_b^{\;Z} Z_t + A_b^{\;v} v_t + A_b^{\;b} \hat{b}_{t-1}^{\,j} + A_b^{\;g} g_t^{\,j} = A_b \bullet (Z_t, v_t, \hat{b}_{t-1}^{\,j}, g_t^{\,j})$

(B.1c) $\hat{q}_{jt}^{\star} = \dfrac{\omega}{1-\omega}[A_\pi^{\;Z} Z_t + A_\pi^{\;v} v_t + A_\pi^{\;b} \hat{b}_{t-1}^{\,j} + A_\pi^{\;g} g_t^{\,j}] = \dfrac{\omega}{1-\omega} A_\pi \bullet (Z_t, v_t, \hat{b}_{t-1}^{\,j}, g_t^{\,j})$

(B.1d) $E_t^j[v_{t+1}^j] = \lambda_v v_t + \lambda_v^g g_t^{\,j}$

(B.1e) $E_t^j[Z_{t+1}] = \lambda_Z Z_t + \lambda_Z^v g_t^{\,i} + \lambda_Z^g g_t^{\,j}$

(B.1f) $E_t^j[g_{t+1}^j] = \lambda_Z g_t + \lambda_Z^v g_t^{\,i}$

One starts by using the consumption and optimal price decision functions but deduce the borrowing function $\hat{b}_t^{\,j}$ from the budget constraint. Hence, write down the two linearized optimal conditions (5a) and (14) to have

$$\hat{c}_t^{\,j} + (\frac{1}{\sigma})[\zeta_\pi \pi_t + \zeta_y \hat{y}_t] = E_t^j(\hat{c}_{t+1}^{\,j}) + (\frac{1}{\sigma}) E_t^j(\pi_{t+1}) + \tau_B \hat{b}_t^{\,j}$$

$$\frac{1-\omega}{\omega}\hat{q}_{jt}^{\star} = -v_t + \kappa(\eta+\sigma)\hat{y}_t + \beta(1-\omega)E_t^j[\hat{q}_{j(t+1)}^{\star} + \pi_{t+1}].$$

These can be written in the linear form implied by (B.1a)-(B.1f) as follows:

$$[A_y \bullet (Z_t, v_t, \hat{b}_{t-1}^{\,j}, g_t^{\,j})] + \frac{\zeta_y}{\sigma}[A_y \bullet (Z_t, v_t, 0, Z_t)] + \frac{\zeta_\pi}{\sigma}[A_\pi \bullet (Z_t, v_t, 0, Z_t)] =$$

$$= A_y \bullet \Big([\lambda_Z Z_t + \lambda_Z^v g_t^{\,i} + \lambda_Z^g g_t^{\,i}], [\lambda_v v_t + \lambda_v^g g_t^{\,j}], A_b \bullet (Z_t, v_t, \hat{b}_{t-1}^{\,j}, g_t^{\,j}), [\lambda_Z g_t + \lambda_Z^v g_t^{\,i}]\Big) +$$

$$+ (\frac{1}{\sigma}) A_\pi \bullet \Big([\lambda_Z Z_t + \lambda_Z^v g_t^{\,i} + \lambda_Z^g g_t^{\,i}], [\lambda_v v_t + \lambda_v^g g_t^{\,j}], 0, [\lambda_Z Z_t + \lambda_Z^v g_t^{\,i} + \lambda_Z^g g_t^{\,i}]\Big) + \tau_B A_b \bullet (Z_t, v_t, \hat{b}_{t-1}^{\,j}, g_t^{\,j})$$

$$A_\pi \bullet (Z_t, v_t, \hat{b}_{t-1}^{\,j}, g_t^{\,j}) - \kappa(\eta+\sigma) A_y \bullet (Z_t, v_t, 0, Z_t) + v_t =$$

$$= \beta\omega A_\pi \bullet \Big([\lambda_Z Z_t + \lambda_Z^v g_t^{\,i} + \lambda_Z^g g_t^{\,i}], [\lambda_v v_t + \lambda_v^g g_t^{\,j}], A_b \bullet (Z_t, v_t, \hat{b}_{t-1}^{\,j}, g_t^{\,j}), [\lambda_Z g_t + \lambda_Z^v g_t^{\,i}]\Big) +$$

$$+ \beta(1-\omega) A_\pi \bullet \Big([\lambda_Z Z_t + \lambda_Z^v g_t^{\,i} + \lambda_Z^g g_t^{\,i}], [\lambda_v v_t + \lambda_v^g g_t^{\,j}], 0, [\lambda_Z Z_t + \lambda_Z^v g_t^{\,i} + \lambda_Z^g g_t^{\,i}]\Big)$$

Given parameters A_b one matches coefficients to have 8 equations, 4 deduced from each of the equations above, in the 8 unknown parameters (A_y, A_π). But to carry this out we need the borrowing function with the penalty on excessive borrowing.

To compute A_b from the budget constraint, the budget to be used is the one deduced from the insurance assumption. That is, the *effective budget which takes into account the transfers*

$$C_t^{\,j} + \frac{M_t^{\,j}}{P_t} + \frac{B_t^{\,j}}{P_t} = (\frac{W_t}{P_t})L_t^{\,j} + [\frac{B_{t-1}^{\,j}(1+r_{t-1}) + M_{t-1}^{\,j}}{P_t}] + \Pi_t \quad , \quad \Pi_t = \int_0^1 [(\frac{p_{jt}}{P_t})^{1-\theta} - \frac{1}{\zeta_t}\frac{W_t}{P_t}(\frac{p_{jt}}{P_t})^{-\theta}] Y_t .$$

This is justified since this is an analysis of the equilibrium and the budget equation above is an equilibrium conditions. Now use the cashless economy assumption and denote by b_t^j the amount of real bonds to deduce

$$C_t^j + b_t^j = (\frac{W_t}{P_t})L_t^j + b_{t-1}^j(1+r_{t-1})\frac{1}{\pi_t} + \int_{S_t}[(\frac{p_{jt}}{P_t})^{1-\theta} - \frac{1}{\zeta_t}\frac{W_t}{P_t}(\frac{p_{jt}}{P_t})^{-\theta}]Y_t dj + \int_{S_t^C}[(\frac{p_{jt-1}}{P_t})^{1-\theta} - \frac{1}{\zeta_t}\frac{W_t}{P_t}(\frac{p_{jt-1}}{P_t})^{-\theta}]Y_t dj$$

By (9) I simplify to

(B.2) $$C_t^j + b_t^j = (\frac{W_t}{P_t})L_t^j + b_{t-1}^j(1+r_{t-1})\frac{1}{\pi_t} + Y_t - \frac{1}{\zeta_t}\frac{W_t}{P_t}Y_t[\int_{S_t}(\frac{p_{jt}}{P_t})^{-\theta}dj + \int_{S_t^C}(\frac{p_{jt-1}}{P_t})^{-\theta}dj].$$

To log linearize (B.2) use (9), (10a)-(10b) to conclude that

$$\hat{c}_t^j + \hat{b}_t^j = \frac{\theta-1}{\theta}[\hat{\ell}_t^j + \hat{w}_t - \hat{p}_t] + \frac{1}{\beta}\hat{b}_{t-1}^j + \hat{y}_t - \frac{\theta-1}{\theta}[-\hat{\zeta}_t + \hat{w}_t - \hat{p}_t + \hat{y}_t]$$

$$= \frac{\theta-1}{\theta}\hat{\ell}_t^j + \frac{1}{\beta}\hat{b}_{t-1}^j + \frac{1}{\theta}\hat{y}_t + \frac{\theta-1}{\theta}\hat{\zeta}_t \quad , \quad \text{where } \hat{b}_t^j = \frac{b_t^j}{\bar{Y}}$$

From (5b -5b') and from the production function we have the sequence

$$\hat{\ell}_t^j = -\frac{\sigma}{\eta}\hat{c}_t^j + \frac{1}{\eta}(\hat{w}_t - \hat{p}_t) = -\frac{\sigma}{\eta}\hat{c}_t^j + \frac{1}{\eta}(\eta\hat{n}_t + \sigma\hat{y}_t) = -\frac{\sigma}{\eta}\hat{c}_t^j + (\hat{y}_t - \hat{\zeta}_t) + \frac{\sigma}{\eta}\hat{y}_t = -\frac{\sigma}{\eta}\hat{c}_t^j - \hat{\zeta}_t + (1+\frac{\sigma}{\eta})\hat{y}_t$$

hence

$$\hat{c}_t^j + \hat{b}_t^j = \frac{\theta-1}{\theta}[-\frac{\sigma}{\eta}\hat{c}_t^j - \hat{\zeta}_t + (1+\frac{\sigma}{\eta})\hat{y}_t] + \frac{1}{\beta}\hat{b}_{t-1}^j + \frac{1}{\theta}\hat{y}_t + \frac{\theta-1}{\theta}\hat{\zeta}_t.$$

Rearranging and solving for $\hat{b}_t^j$ the borrowing function is then

$$\hat{b}_t^j = \frac{1}{\beta}\hat{b}_{t-1}^j + [1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}](\hat{y}_t - \hat{c}_t^j).$$

Return now to (B.1a) - (B.1b) and by matching coefficients deduce that

(B.3) $$A_b^Z = A_y^g[1+\frac{\theta-1}{\theta}\frac{\sigma}{\eta}] \ , \ A_b^v = 0 \ , \ A_b^b = \frac{1}{\beta} - A_y^b[1+\frac{\theta-1}{\theta}\frac{\sigma}{\eta}] \ , \ A_b^g = -A_y^g[1+\frac{\theta-1}{\theta}\frac{\sigma}{\eta}].$$

This last computation was based on a comparison of the following two functions

$$\hat{c}_t^j = A_y^Z Z_t + A_y^v v_t + A_y^b \hat{b}_{t-1}^j + A_y^g g_t^j = A_y \bullet (Z_t, v_t, \hat{b}_{t-1}^j, g_t^j)$$

$$\hat{y}_t = A_y^Z Z_t + A_y^v v_t + A_y^b 0 + A_y^g Z_t \equiv A_y \bullet (Z_t, v_t, 0, Z_t).$$

To see why, note that given the individual decision functions above, aggregate functions are deduced from them by market clearing conditions which are

$$\int_0^1 \hat{c}_t^j dj = \hat{y}_t \ , \ \int_0^1 \hat{b}_t^j dj = 0, \ Z_t = \int_0^1 g_t^j dj \ \text{ and } \int_0^1 \hat{q}_{jt}^\star = \frac{\omega}{1-\omega}\pi_t .$$

By (B.1a)-(B.1c)

(B.4a) $$\hat{y}_t = A_y^Z Z_t + A_y^v v_t + A_y^b 0 + A_y^g Z_t \equiv A_y \bullet (Z_t, v_t, 0, Z_t)$$

(B.4b) $$\hat{\pi}_t = A_\pi^Z Z_t + A_\pi^v v_t + A_\pi^b 0 + A_\pi^g Z_t \equiv A_\pi \bullet (Z_t, v_t, 0, Z_t)$$

(B.4c) $$\hat{q}_t = \frac{\omega}{1-\omega}[A_\pi^Z Z_t + A_\pi^v v_t + A_\pi^b 0 + A_\pi^g Z_t] \equiv \frac{\omega}{1-\omega}A_\pi \bullet (Z_t, v_t, 0, Z_t).$$

A Note on Simulations

Once matching of parameters is completed and the values of (A_y, A_π, A_b) are determined, the solutions (B.4a)- (B.4c) can be used to simulate the system of structural equations and law of motion. This is a simple procedure in which one uses (B.4a)-(B.4c) to select an initial condition given some (Z_0, u_0). Next one simulates a system like (31a)-(31b) to obtain a sequence of (Z_t, v_t) which is then inserted into (B.4a)- (B.4c) to compute the implied values of the aggregate endogenous variables.

The method outlined is simpler than the standard procedures used to simulate a Blanchard-Kahn type of a system. However, a standard simulation can be carried out with off-the-shelf programs for simulating forward looking system of difference equations. Such a standard procedure can be used once the constants (B_y, B_π) are computed using the parameters computed from the micro economic equilibrium as outlined above. The reader can check that the results are identically the same in both methods. Naturally, the simplicity of the first method results from the general equilibrium approach taken in the text to determine individual decision functions and their parameters in the log linearized economy as outlined in this appendix. However, if the micro economic equilibrium becomes more complicated and entails, for example, an infinite number of state variables, such procedure may not be feasible.